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Nearly Uniform Convexity of a Nörlund-Musielak-Orlicz Sequence Space

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Abstract

In this paper we define the Nörlund - Musielak - Orlicz sequence space $\aleph(\mathcal{M})$ which is a generalisation of the Cesàro sequence space ces_p. We exhibit some geometric properties for this space.

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1. INTRODUCTION

Let $(X, ||.||)$ be a real Banach space. By $B(X)$ and $S(X)$, we denote the closed unit ball and the unit sphere in X, respectively.

A norm $||.||$ is called uniformly convex (UC) (cf[3]) if, for each $\epsilon > 0$, there is $\delta > 0$ such that , for $x, y \in S(X)$, the inequality $||x - y|| > \epsilon$ implies

$$(1) \qquad ||\frac{1}{2}(x + y)|| < 1 - \delta.$$

For any $x \notin B(X)$, the drop property determined by x is the set

$$D(x, B(X)) = conv(\{x\} \cup B(X)).$$

A Banach space X has the drop property (see[6,15]) (write (D)) if for every closed set C disjoint with $B(X)$ there exists an element $x \in C$ such that

$$D(x, B(X)) \cap C = \{x\}.$$

A Banach space X is said to have the Kadak- Klee (or property(H)) (see[12]) if every weakly convergent sequence on the unit sphere is convergent in norm.

Recall that a sequence (x_n) is said to be an ϵ - separated sequence if for some $\epsilon > 0$,

$$sep(x_n) = \inf\{||x_n - x_m|| : n \neq m\} > \epsilon.$$

A Banach space X is said to be nearly uniformly convex (write (NUC)) if for every $\epsilon > 0$ there exists $\delta \in (0,1)$ such that for every sequence $(x_n) \subseteq B(X)$ with $sep(x_n) > \epsilon$, we have

$$conv(x_n) \cap ((1-\delta)B(X)) \neq \emptyset.$$

Huff [9] proved that every (NUC) Banach space is reflexive and it has proprty (H).

A Banach space X is said to be k-nearly uniformly convex (write (k-NUC)) if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every sequence $(x_n) \subset B(X)$ with $sep(x_n) \geq \epsilon$ there are $n_1, n_2, \cdots, n_k \in \mathbb{N}$ such that

$$||\frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k}|| < 1 - \delta.$$

Clearly $k - NUC$ Banach spaces are NUC but converse is not true in general (see[11]).

A Banach space X is said to be fully k-rotund (write kR) (See[8]) for every sequence $(x_n) \subset B(X)$,

$$||x_{n_1} + x_{n_2} + \cdots + x_{n_k}|| \to k \quad as \quad n_1, n_2, \cdots, n_k \to \infty$$

implies (x_n) is convergent.

It is well known that $UC \Rightarrow kR \Rightarrow (k+1)R$, and kR spaces are reflexive and rotund, and it is easy to see that $k - NUC \Rightarrow kR$.

A Banach space X is said to have the Banach-Saks property (cf [1]) if every bounded sequence (x_n) in X admits a subsequence (z_n) such that the sequence $(t_k(z))$ is convergent in norm in X (see [1]), where

$$t_k(z) = \frac{1}{k}(z_1 + z_2 + \cdots + z_k).$$

Every Banach space X with the Banach-Saks property is reflexive and the converse is not true (see[7]). Kakutani [10] proved that any uniformly convex

Banach space X has the Banach-Saks property. Moreover , he also proved that if X is a reflexive Banach space and there is $\theta \in (0, 2)$ such that for every sequence (x_n) in $S(X)$ weakly convergent to zero, there are $n_1, n_2 \in I\!N$ satisfying $||x_{n_1} + x_{n_2}|| < \theta$, then X has the Banach-Saks property.

For a sequence $(x_n) \subset X$, we define

$$A((x_n)) = \lim_{n\to\infty} \inf\{||x_i + x_j|| : i, j \geq n, i \neq j\}.$$

Let X be a real vector space . A functional $\varrho : X \to [0, \infty]$ is called a modular if

(*i*) $\varrho(x) = 0$ if and only if $x = \theta$;

(*ii*) $\varrho(\alpha x) = \varrho(x)$ for all scalar α *with* $|\alpha| = 1$;

(*iii*) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$, for all x, y $\in X$ *and* α, $\beta \geq 0$ *with* $\alpha + \beta = 1$.

The modular ϱ is called convex if

(*iv*) $\varrho(\alpha x + \beta y) \leq \alpha\varrho(x) + \beta\varrho(y)$, for all x , $y \in X$ and α, $\beta \geq 0$ with $\alpha + \beta = 1$.

If ϱ is a modular in X , we define

$$X_\varrho = \{x \in X : \lim_{\lambda\to 0^+} \varrho(\lambda x) = 0\}$$

and

$$X^*_\varrho = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

It is clear that $X_\varrho \subseteq X^*_\varrho$. If ϱ is a convex modular, for $x \in X_\varrho$ we define

$$||x|| = \inf\{\lambda > 0 : \varrho(\frac{x}{\lambda}) \leq 1\} \tag{1.1}$$

It is known that if ϱ is a convex modular on X , then $X_\varrho = X^*_\varrho$ and $||.||$ is a norm on X_ϱ for which it is a Banach space. The norm $||.||$ defined as in (1.1) is cllled the Luxemburg norm.

The following known results gave some relationships between the modular ϱ and the Luxemburg norm $||.||$ on X_ϱ.

Theorem 1.1 Let ϱ be a convex modular on X and let $x \in X_\varrho$ and $(x^{(n)})$ be a sequence in X_ϱ. Then $||x^{(n)} - x|| \to 0$ as $n \to \infty$ if and only if $\varrho(\lambda(x^{(n)} - x)) \to 0$ as $n \to \infty$ for every $\lambda > 0$.

Proof. See [13, Theorem 1.3].

A map $\phi : R \to [0, \infty]$ is said to be an Orlicz function if ϕ vanishes only at 0, and ϕ is even and convex.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak - Orlicz function. In addition, a Musielak-Orlicz function $N = (N_k)$ is called a complementary function of a Musielak - Orlicz function M if

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \quad k = 1, 2, \cdots$$

For a given Musielak - Orlicz function $\mathcal{M}$, the Musielak - Orlicz sequence space $l_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows :

$$l_{\mathcal{M}} := \{x \in l^0 : I_M(cx) < \infty \ \text{ for } \ \text{some} \ \ c > 0\},$$

$$h_{\mathcal{M}} := \{x \in l^0 : I_{\mathcal{M}}(cx) < \infty \ \text{ for } \ \text{all} \ \ c > 0\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in l_{\mathcal{M}}.$$

We consider $l_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\{k > 0 : I_{\mathcal{M}}(\frac{x}{k}) \leq 1\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\{\frac{1}{k}(1 + I_{\mathcal{M}}(kx)) : k > 0\}.$$

Now we define the Nörlund-sequence space introduced in [2] as follows;

$$\aleph_p = \{x = (x_k) : \sum_{n=0}^{\infty} |\frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k|^p < \infty, \quad 1 \leq p < \infty\},$$

where $A_n = \sum_{k=0}^{n} a_k$,
which is a Banach space with usual norm

$$||x||_p = \left(\sum_{n=0}^{\infty} |\frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k|^p \right)^{1/p}.$$

Let $\mathcal{M} = (M_k)$ be a Musielak - Orlicz function and $a = (a_k)$ be a sequence of positive real numbers. The Nörlund-Musielak-Orlicz sequence space is defined by

$$\aleph(\mathcal{M}) := \{x \in l^0 : \varrho_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\},$$

where $\varrho_{\mathcal{M}}$ is a convex modular defined by

$$\varrho_{\mathcal{M}}(x) = \sum_{n=0}^{\infty} M_n(|\frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k|),$$

and $A_n = \sum_{k=0}^{n} a_k$.

For $M_k(x) = |x|^p$, $\aleph(\mathcal{M})$ is reduced to the Nörlund sequence space defined by Wang [2].

We consider $\aleph(\mathcal{M})$ equipped with the Luxemburg norm

$$||x|| = \inf\{\lambda > 0 : \varrho_{\mathcal{M}}(\frac{x}{\lambda}) \leq 1\}$$

under which it is a Banach space. We define the subspace $S\aleph(\mathcal{M})$ of $\aleph(\mathcal{M})$ by

$$S\aleph(\mathcal{M}) := \{x \in l^0 : \varrho_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\}.$$

A Musielak - Orlicz function $\mathcal{M}$ satisfies the δ_2 condition ($\mathcal{M} \in \delta_2$ for short) if there exist constants $K \geq 2, u_0 > 0$ and a sequence (c_k) of positive numbers such that $\sum_{k=1}^{\infty} c_k < \infty$ and the inequality

$$M_k(2u) \leq K M_k(u) + c_k$$

holds for every $k \in \mathbb{N}$ and $u \in R$ satisfying $M_k(u) \leq u_0$.

If $\mathcal{M} \in \delta_2$ and $N \in \delta_2$, then we write $\mathcal{M} \in \delta_2 \cap \delta_2^*$. It is known that $l_{\mathcal{M}} = h_{\mathcal{M}}$ if and only if $\mathcal{M} \in \delta_2$ (see[14]). Moreover we say that a Musielak-Orlicz function $\mathcal{M} = (M_k)$ satiisfies conditions $(*)$ if for any $\epsilon \in (0, 1)$, there exists $\delta > 0$ such that

$$M_k((1+\delta)u) \leq 1 \text{ whenever } M_k(u) \leq 1 - \epsilon \text{ for all } k \in \mathbb{N} \text{ and } u \in R.$$

In this paper we defined a new sequence space $\aleph(\mathcal{M})$, which is a generalization of the Norlund sequence space by using a Musielak - Orlicz function , and we call the space $\aleph(\mathcal{M})$, Nörlund- Musielak-Orlicz sequenc space. We shall show that if $\mathcal{M} \in \delta_2 \cap \delta^*$ and $\mathcal{M}$ satifies the condition (*), then $\aleph(\mathcal{M})$ is $k - NUC$, so it has Banach-Saks property.

2. AUXILIARY RESULTS

Proposition 2.1. For any $x \in \aleph(\mathcal{M})$, we have

(a) $||x|| \leq 1$, then $\varrho_{\mathcal{M}}(x) \leq ||x||$, and

(b) $||x|| > 1$, then $\varrho_{\mathcal{M}}(x) \geq ||x||$.

Proof. (a) If $x = 0$, then the inequality holds. Let $x \neq 0$. Then by the definition of $||.||$, there is a sequence (ϵ_n) such that $\epsilon_n \downarrow ||x||$ such that $\varrho_{\mathcal{M}}(\frac{x}{\epsilon_n}) \leq 1$. This implies $\varrho_{\mathcal{M}}(\frac{x}{||x||}) \leq 1$. Since $\varrho_{\mathcal{M}}$ is convex, we have $\varrho_{\mathcal{M}}(x) \leq ||x||\varrho_{\mathcal{M}}(\frac{x}{||x||}) \leq ||x||$.

(b) Let $||x|| > 1$. Then for $\epsilon \in (0, \frac{||x||-1}{||x||})$, we have $(1-\epsilon)||x|| > 1$. By convexity of $\varrho_{\mathcal{M}}$, we have $1 < \varrho_{\mathcal{M}}(\frac{x}{(1-\epsilon)||x||}) \leq \frac{\varrho_{\mathcal{M}}(x)}{(1-\epsilon)||x||}$, so that $(1-\epsilon)||x|| < \varrho_{\mathcal{M}}(x)$. By taking $\epsilon \to 0$, we have $\varrho_{\mathcal{M}}(x) \geq ||x||$.

The following result is directly obtained form Proposition 2.1(a).

Corollary 2.2. If (x_n) is a sequence $\aleph(\mathcal{M})$ such that $x_n \to 0$ as $n \to \infty$, then $\varrho_M(x_n) \to 0$ as $n \to \infty$.

Proposition 2.3. If a Musielak - Orlicz function $\mathcal{M} = (M_k) \in \delta_2$, then $S\aleph(\mathcal{M}) = \aleph(\mathcal{M})$.

Proof . If $x \in \aleph(\mathcal{M})$, then the sequence $a = (a(k))$, defined by $a(k) = |\frac{1}{A_n}\sum_{k=0}^{n} a_{n-k}x(k)|$ for all $k \in \mathbb{N}$ is in $l_{\mathcal{M}}$. By $\mathcal{M} \in \delta_2$ we have that $l_{\mathcal{M}} = h_{\mathcal{M}}$. This implies that $\varrho_{\mathcal{M}}(\lambda x) = I_{\mathcal{M}}(\lambda a) < \infty$ for all $\lambda > 0$ hence $x \in \aleph(\mathcal{M})$.

Proposition 2.4. If the Musielak-Orlicz function $\mathcal{M} = (M_k) \in \delta_2$, then

(1) $||x|| = 1 \Leftrightarrow \varrho_{\mathcal{M}}(x) = 1$,

(2) for every $\epsilon > 0$, there exists a $\delta > 0$ such that $||x|| < 1 - \delta$ whenever $\varrho_{\mathcal{M}}(x) < 1 - \epsilon$.

Proof. See [16]

Proposition 2.5. If the Musielak-Orlicz function $\mathcal{M} = (M_k)$ satiisfies conditions $(*)$ and $\mathcal{M} \in \delta_2$, then
(i) for every $\epsilon > 0$ and $c > 0$, there exists a $\delta > 0$ such that for any $x, y \in \aleph(\mathcal{M})$, we have
$|\varrho_{\mathcal{M}}(x+y) - \varrho_{\mathcal{M}}(x)| < \epsilon$ whenever $\varrho_{\mathcal{M}}(x) \leq c$ and $\varrho_{\mathcal{M}}(x) \leq \delta$, and
(ii) for every $\epsilon > 0$, there exists a $\delta > 0$ such that $||x|| > 1 + \delta$ whenever $\varrho_{\mathcal{M}}(x) > 1 + \epsilon$.

Proof. Let $\epsilon > 0$ and $c > 0$. By [5, Lemma 8], there exists a $\delta > 0$ such that for any $a, b \in l_{\mathcal{M}}$, we have

$$|I_{\mathcal{M}}(a+b) - I_{\mathcal{M}}(a)| < \epsilon \tag{2.1}$$

whenever $I_{\mathcal{M}}(a) \leq c$ and $I_{\mathcal{M}}(b) \leq \delta$. For each $i \in I\!\!N$, let

$$s(i) = \begin{cases} sgn(x(i) + y(i)) & , \text{ if } \quad x(i) + y(i) \neq 0, \\ 1 & , \text{ if } \quad x(i) + y(i) \neq 0. \end{cases}$$

Then we have

$$\varrho_{\mathcal{M}}(x+y) = \sum_{k=1}^{\infty} M_k(|\frac{1}{A_n} \sum_{i=0}^{n} a_{n-i}x(i) + a_{n-i}y(i)|)$$

$$= \sum_{k=1}^{\infty} M_k(|\frac{1}{A_n} \sum_{i=0}^{n} s(i)a_{n-i}x(i)| + |\frac{1}{A_n} \sum_{i=0}^{n} s(i)a_{n-i}y(i)|. \tag{2.2}$$

Let $a(k) = |\frac{1}{A_n} \sum\limits_{i=0}^{n} s(i)a_{n-i}x(i)|$ and $b(k) = |\frac{1}{A_n} \sum\limits_{i=0}^{n} s(i)a_{n-i}y(i)|$ for all $n \in I\!\!N$. Then and $a = (a(k)) \in l_{\mathcal{M}}$ and $b = (b(k)) \in l_{\mathcal{M}}$, and from (2.2), we have

$$\varrho_{\mathcal{M}}(x+y) = I_{\mathcal{M}}(a+b), I_{\mathcal{M}}(a) \leq \varrho_{\mathcal{M}}(x) \;\; and \;\; I_{\mathcal{M}}(b) \leq \varrho_{\mathcal{M}}(y).$$

Let $x, y \in \aleph(\mathcal{M})$ then $\varrho_{\mathcal{M}}(x) \leq c$ and $\varrho_{\mathcal{M}}(y) \leq \delta$. Thus $I_m(a) \leq c$ and $I_{\mathcal{M}}(b) \leq \delta$. By (2.1) we have

$$\varrho_{\mathcal{M}}(x+y) - \varrho_{\mathcal{M}}(x) \leq I_m(a+b) - I_{\mathcal{M}}(a) < \epsilon,$$

that is,

$$\varrho_{\mathcal{M}}(x+y) < \varrho_{\mathcal{M}}(x) + \epsilon \tag{2.3}$$

Now we need to show that

$$\varrho_{\mathcal{M}}(x) < \varrho_{\mathcal{M}}(x+y) + \epsilon. \tag{2.4}$$

Let for all $i \in \mathbb{N}$,

$$s(i) = \begin{cases} sgn(x(i)) & , \text{ if } \quad x(i) \neq 0, \\ 1 & , \text{ if } \quad x(i) \neq 0. \end{cases}$$

Then we have

$$\varrho_{\mathcal{M}}(x) = \varrho_{\mathcal{M}}((x+y) + (-y)) =$$
$$= \sum_{n=0}^{\infty} M_n \left(|\frac{1}{A_n} \sum_{i=0}^{n} a_{n-i}(x(i) + y(i)) + (-y(i))| \right)$$
$$= \sum_{n=0}^{\infty} M_n \left(|\frac{1}{A_n} \sum_{i=0}^{n} s(i) a_{n-i}(x(i) + y(i))| + |\frac{1}{A_n} \sum_{i=0}^{n} s(i)(-a_{n-i} y(i))| \right).$$

Put $a(n) = |\frac{1}{A_n} \sum_{i=0}^{n} s(i) a_{n-i}(x(i)+y(i))|$ and $b(n) = |\frac{1}{A_n} \sum_{i=0}^{n} s(i)(-a_{n-i} y(i))|$ for all $n \in \mathbb{N}$. It is clear that $a = (a(k)) \in l_{\mathcal{M}}$ and $b = (b(k)) \in l_{\mathcal{M}}$, and $\varrho_{\mathcal{M}}(x+y) = I_{\mathcal{M}}(a+b)$, $I_{\mathcal{M}}(a) \leq \varrho_{\mathcal{M}}(x+y)$ and $I_{\mathcal{M}}(b) \leq \varrho_{\mathcal{M}}(y)$. Hence we have $I_{\mathcal{M}}(a+b) \leq c$ and $I_{\mathcal{M}}(-b) \leq \delta$. By (2.3), we have

$$|I_{\mathcal{M}}(a+b) - I_{\mathcal{M}}(a)| = |I_{\mathcal{M}}(a) - I_{\mathcal{M}}(a+b)| \tag{2.5}$$

$$= |I_{\mathcal{M}}(a+b) + (-b)) - I_{\mathcal{M}}(a+b)| < \epsilon. \tag{2.6}$$

This implies that

$$\varrho_{\mathcal{M}}(x) - \varrho_{\mathcal{M}}(x+y) \leq I_{\mathcal{M}}(a+b) - I_{\mathcal{M}}(a) < \epsilon,$$

hence

$$\varrho_{\mathcal{M}}(x) < \varrho_{\mathcal{M}}(x+y) + \epsilon.$$

Thus (2.4) holds. Therefore (i) is obtained by (2.3) and (2.4).

(ii) For any given $\epsilon > 0$ and by (i), there exists $\delta \in (0,1)$ such that

$$\varrho_{\mathcal{M}}(u) \leq 1, \varrho_{\mathcal{M}}(v) \leq \delta$$

implies that

$$\varrho_{\mathcal{M}}(u+v) \leq \varrho_{\mathcal{M}}(u) + \epsilon. \tag{2.7}$$

If $||x|| \leq 1+\delta$, then $\varrho_M(\frac{x}{1+\delta}) \leq 1$ and

$$\varrho_{\mathcal{M}}(\frac{\delta x}{1+\delta}) \leq \delta \varrho_{\mathcal{M}}(\frac{x}{1+\delta}) \leq \delta.$$

By (2.5) we have

$$\varrho_{\mathcal{M}}(x) = \varrho_{\mathcal{M}}(\frac{x}{1+\delta} + \frac{\delta x}{1+\delta}) \leq \frac{x}{1+\delta} + \epsilon \leq 1+\epsilon.$$

Hence (ii) is satisfied.

Proposition 2.6. If the Musielak-Orlicz function $\mathcal{M} = (M_k)$ satisfies conditions $(*)$ and $\mathcal{M} \in \delta_2$, then for any sequence $(x_n) \subset \aleph(\mathcal{M})$, $||x_n|| \to 1$ implies $\varrho_M(x_n) \to 1$.

Proof. Let $\varrho_M(x_n) \not\to 1$ as $n \to \infty$. Assume that therer exists $\epsilon_0 > 0$ such that $|\varrho_M(x_n) - 1 > \epsilon_0$ for all $n \in I\!N$. If $\varrho_M(x_n) - 1 > \epsilon_0$, then $\varrho_M(x_n) > 1 + \epsilon_0$ by Proposition 2.5(ii), there exists $\delta > 0$ such that $||x_n|| > 1+\delta$. $\varrho_M(x_n) - 1 < -\epsilon_0$, then $\varrho_M(x_n) < 1 - \epsilon_0$, by Proposition 2.4(2), there exists a $\delta^{'} > 0$ such that $||x_n|| < 1 - \delta^{'}$. Hence $||x_n|| \not\to 1$ as $n \to \infty$.

Proposition 2.7. In Nörlund-Musielak - Orlicz sequence space.If a Musielak-Orlicz function $\mathcal{M} = (M_k)$ satiisfies conditions $(*)$ and $\mathcal{M} \in \delta_2$, then the norm convergence and modular convergence coincide.

Proof.(See [16])

3. MAIN RESULTS

Theorem 3.1. If the Musielak-Orlicz function $\mathcal{M} = (M_k)$ satisfies conditions $(*)$ and $\mathcal{M} \in \delta_2 \cap \delta_2^*$, then $\aleph(\mathcal{M})$ is $k - NUC$.

Proof. Let $\epsilon > 0$ be given and sequence $(x_n) \subseteq B(\aleph(\mathcal{M}))$ with $sep\{(x_n)\} > \epsilon$. For $\mathcal{M} \in \delta_2$ there exists a $\delta > 0$ such that

$$\inf\{\varrho_{\mathcal{M}}(\frac{x_n - x_m}{2}) : n \neq m\} \geq \delta.$$

Now we will show that for any $j \in \mathbb{N}$ there exists $n_j \in \mathbb{N}$ such that

$$\sum_{i=j}^{\infty} M_i(|\frac{1}{A_i}\sum_{l=0}^{i} a_{i-l}x_{n_j}(l)|) \geq \frac{\delta}{3} \tag{3.1}$$

otherwise, there exists a $j_0 \in \mathbb{N}$ such that

$$\sum_{i=j_0}^{\infty} M_i(|\frac{1}{A_i}\sum_{l=0}^{i} a_{i-l}x_{n_j}(l)|) < \frac{\delta}{3}$$

for any $j \in \mathbb{N}$. Put $y_n = (x_n(1), x_n(2), x_n(3), \cdots, x_n(j_0), 0, 0, \cdots)$ for $n \in \mathbb{N}$. Then there exists a subsequence $(y_{n_l}) \subset (y_n)$ such that

$$\varrho_{\mathcal{M}}(\frac{y_{n_l} - y_{n_j}}{2}) < \frac{\delta}{3} \qquad \text{for any} \quad l \neq j.$$

Hence

$$\varrho_{\mathcal{M}}(\frac{x_{n_l} - x_{n_j}}{2}) = \varrho_{\mathcal{M}}(\frac{\sum_{m=1}^{j_0}(x_{n_l}(m) - x_{n_j}(m))e_m}{2})$$

$$+\varrho_{\mathcal{M}}(\frac{\sum_{m=j_0+1}^{j_0}(x_{n_l}(m) - x_{n_j}(m))e_m}{2})$$

$$\leq \varrho_{\mathcal{M}}(\frac{\sum_{m=1}^{j_0}(x_{n_l}(m) - x_{n_j}(m))e_m}{2})$$

$$+\frac{1}{2}\sum_{i=j_0+1}^{\infty} M_i(|\frac{1}{A_i}\sum_{m=0}^{i} a_{i-m}x_{n_l}(m)|)$$

$$+\frac{1}{2}\sum_{i=j_0+1}^{\infty} M_i(|\frac{1}{A_i}\sum_{m=0}^{i} a_{i-m}x_{n_j}(m)|)$$

$$= \varrho_{\mathcal{M}}(\frac{y_{n_l} - y_{n_j}}{2}) + \frac{1}{2}\sum_{i=j_0+1}^{\infty} M_i(|\frac{1}{A_i}\sum_{m=0}^{i} a_{i-m}x_{n_l}(m)|)$$

$$+\frac{1}{2}\sum_{i=j_0+1}^{\infty} M_i(|\frac{1}{A_i}\sum_{m=0}^{i} a_{i-m}x_{n_j}(m)|)$$

$$< \frac{\delta}{3} + \frac{\delta}{6} + \frac{\delta}{6} = \frac{2\delta}{3} < \delta,$$

which is a contradiction. Hence (2.7) is correct. Since $N \in \delta_2$, there exists $\theta \in (0,1)$ and a sequence $(h_i) \in R^+$ with $\sum_{i=1}^{\infty} M_i(h_i) < \infty$ such that

$$M_i(\frac{u}{k}) \leq \frac{1-\theta}{k} M_i(u)$$

holds for every $i \in \mathbb{N}$ and u satisfies $M_i(h_i) \leq M_i(u) \leq 1$. Using $\mathcal{M} \in \delta_2$ again , there exists $\delta_1 > 0$ such that

$$|\varrho_{\mathcal{M}}(x+y) - \varrho_{\mathcal{M}}(x)| < \frac{\theta\delta}{12k}$$

whenever $\varrho_{\mathcal{M}}(x) \leq 1$ and $\varrho_{\mathcal{M}}(y) < \delta_1$. Take $n_1 < n_2 < \cdots < n_{k-1} \in \mathbb{N}$. Note that

$$\varrho_{\mathcal{M}}(\frac{x_{n_1} + x_{n_2} + x_{n_3} + \cdots + x_{n_{k-1}}}{k}) < \infty$$

and $\varrho_{\mathcal{M}}(x_{n_l}) < \infty$ for $l = 1, 2, \cdots, k-1$. There exists a $j_0 \in \mathbb{N}$ such that

(3.2)
$$\sum_{i=j_0+1}^{\infty} M_i(|\frac{1}{A_i}\sum_{l=0}^{i} \frac{a_{i-1}x_{n_1}(l) + a_{i-2}x_{n_2}(l) + \cdots + a_{i-(k-1)}x_{n_{k-1}}(l)}{k}|) < \delta_1,$$

(3.3)
$$\sum_{i=j_0+1}^{\infty} M_i(|\frac{1}{A_i}\sum_{l=0}^{i} a_{i-l}x_{n_j}(l)|) < \frac{\delta}{3}, \qquad (j = 1, 2, \cdots, k-1)$$

(3.4)
$$\sum_{i=j_0+1}^{\infty} M_i(h_i) < \frac{\theta\delta}{12k}.$$

By using (3.1) there $\exists$ a $n_k \in \mathbb{N}$ such that

$$\sum_{i=j_0+1}^{\infty} M_i(|\frac{1}{A_i}\sum_{l=0}^{i} a_{i-l}x_{n_k}(l)|) \geq \frac{\delta}{3}. \tag{3.5}$$

From (2.3),(3.1),(3.2)and (3.3) we have

$$\varrho_{\mathcal{M}}(\frac{x_{n_1}+x_{n_2}+x_{n_3}+\cdots+x_{n_k}}{k})$$

$$= \sum_{i=1}^{j_0} M_i(|\frac{1}{A_i}\sum_{l=0}^{i}\frac{a_{i-1}x_{n_1}(l)+a_{i-2}x_{n_2}(l)+\cdots+a_{i-k}x_{n_k}(l)}{k}|)$$

$$+\sum_{i=j_0+1}^{\infty} M_i(|\frac{1}{A_i}\sum_{l=0}^{i}\frac{a_{i-1}x_{n_1}(l)+a_{i-2}x_{n_2}(l)+\cdots+a_{i-k}x_{n_k}(l)}{k}|)$$

$$\leq \frac{1}{A_k}\sum_{j=0}^{k}\sum_{i=1}^{j_0} M_i(|\frac{1}{A_i}\sum_{l=0}^{i} a_{i-l}x_{n_j}(l)|)$$

$$+\sum_{i=j_0+1}^{\infty} M_i(|\frac{1}{A_i}\sum_{l=0}^{i}\frac{a_{i-l}x_{n_k}(l)}{k}|)+\frac{\delta\theta}{12k}$$

$$\leq \frac{1}{A_k}\sum_{j=0}^{k}\sum_{i=0}^{j_0} M_i(|\frac{1}{A_i}\sum_{l=0}^{i} a_{i-l}x_{n_j}(l)|)$$

$$+\frac{1-\theta}{k}\sum_{i=j_0+1}^{\infty} M_i(|\frac{1}{A_i}\sum_{l=0}^{i} a_{i-l}x_{n_k}(l)|)+\sum_{i=j_0+1}^{\infty} M_i(h_i)+\frac{\delta\theta}{12k}$$

$$\leq \frac{1}{A_k}\sum_{j=0}^{k}\sum_{i=1}^{j_0} M_i(|\frac{1}{A_i}\sum_{l=0}^{i} a_{i-l}x_{n_j}(l)|)$$

$$-\frac{\theta}{k}\sum_{i=j_0+1}^{\infty} M_i(|\frac{1}{A_i}\sum_{l=0}^{i} a_{i-l}x_{n_k}(l)|)+\frac{\delta\theta}{6k}$$

$$\leq 1-\frac{\delta\theta}{3k}+\frac{\delta\theta}{6k}=1-\frac{\delta\theta}{6k}.$$

Since $\mathcal{M} \in \delta_2$ and satisfies (*) - condition, by Proposition 2.4(2) there is $\gamma \in (0,1)$ such that

$$||\frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k}|| < 1 - \gamma.$$

Thus $\aleph(\mathcal{M})$ is $k - NUC$.

Corollary 2.9. If the Musielak-Orlicz function $\mathcal{M} = (M_k)$ satiisfies conditions $(*)$ and $\mathcal{M} \in \delta_2 \cap \delta_2^*$, then

(1) $\aleph(\mathcal{M})$ has the Banach-Saks property i.e. $\aleph(\mathcal{M})$ is reflexive and it has weak Banach -Saks property.

(2) $\aleph(\mathcal{M})$ is NUC.

We know that $k - NUC \Rightarrow kR \Rightarrow R$ and Rfx and $k - NUC \Rightarrow (NUC) \Rightarrow property(H)$ and Rfx, where Rfx denotes for reflexivity.

By Theorem 3.1 We have the following results.

Corollary 3.3. For $1 < p < \infty$, the space $\aleph_p$ is $k - NUC$.

Corollary 3.4. For $1 < p < \infty$, the space $\aleph_p$ is kR and (NUC).

Corollary 3.5. For $1 < p < \infty$, the space $\aleph_p$ has the drop property.

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