International Mathematical Forum, 3, 2008, no. 40, 1991 - 1999

A Note on Bernardi's Integral Operators of Certain Classes of Analytic Functions

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Abstract

Let $S^*(\alpha)$ denote the class of functions f analytic in the open unit disc $\mathcal{D} = \{z \in \mathbb{C}; |z| < 1\}$, normalized and satisfying

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1} \right| < \alpha, \qquad z \in \mathcal{D}.$$

Making use of the following integral operator, that is

$$\mathcal{F} = I_{\beta,\gamma} f(z) = \left[\frac{\beta + \gamma}{z^\gamma} \int_0^z f^\beta(t) t^{\gamma - 1} dt \right]^{1/\beta},$$

where β, $\gamma \in \mathbb{C}$, with $\beta \neq 0$, $\beta > 0$, $\gamma \geq 0$, $Re(\beta + \gamma) > 0$, we determine δ such that whenever $f \in S^*(\alpha)$ then $\mathcal{F} \in S^*(\delta)$. Also in this paper, a similar problem for the class $R_\beta(\alpha)$ of all analytic functions satisfying

$$R_\beta(\alpha) = \left| \frac{\frac{f'(z)}{(f(z))^{1-\beta}} - 1}{\frac{f'(z)}{(f(z))^{1-\beta}} + 1} \right| < \alpha, \qquad z \in \mathcal{D}$$

is investigated. Thus generalise some known results.

Mathematics Subject Classification: 30C45

Keywords: starlike of order α, Bernardi's integral operator

1 Introduction

Let $H(\mathcal{D})$ denotes the class of functions f analytic in the open unit disc $\mathcal{D} = \{z \in \mathbb{C}; |z| < 1\}$ and $S = \{f \in H(\mathcal{D}) : f(0) = 0 = f'(0) - 1\}$. Also, let α be a given real number $0 < \alpha \leq 1$ and define that $f \in S^*(\alpha)$ if

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1} \right| < \alpha, \qquad z \in \mathcal{D}.$$

$S^*(1)$ is the well-known class S^* of starlike functions with respect to the origin, that is

$$\text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad (z \in \mathcal{D}).$$

1.1 The class of Caratheodory Functions

Let $p(z) = 1 + p_1 z + p_2 z + \ldots$ be a function regular and analytic in $\mathcal{D}$ and satisfies the conditions $p(0) = 1$ and Re $p(z) > 0$. Then this type of function is known as *Caratheodory* functions and denoted by $\mathcal{P}$. If we use the subordination principle we have

$$p(z) \in p \quad \text{if and only if} \quad p(z) \prec \frac{1+z}{1-z}. \tag{1.1}$$

1.2 The class of Janowski Functions

Let $p(z) = 1 + b_1 z + b_2 z + \ldots$ be regular and analytic in $\mathcal{D}$ and satisfies the condition

$$p(0) = 1 \quad \text{Re}p(z) > 0, \quad p(z) \prec \frac{1+Az}{1-Bz}, -1 < A < 1, \quad -1 \leq B < A \tag{1.2}$$

then this function is known as *Janowski* functions and denoted by $\mathcal{P}(A, B)$. Geometrically, $p(z)$ is in $\mathcal{P}(A, B)$ if and only if $p(0) = 1$ and $p(\mathcal{D})$ inside the open disc centered on the real axis with diameter end points:

$$p(-1) = \frac{1-A}{1-B} \quad \text{and} \quad p(1) = \frac{1+A}{1+B}.$$

Special selections of A and B lead to familiar sets defined by inequalities under the conditions $p(0) = 1$, $M > \frac{1}{2}$ and $0 \leq \alpha < 1$, we have

1. $p(-1, 1) = p$ is the set defined by Re$p(z) > 0$ (Caratheodory's Class)

2. $p(1-2\alpha,-1)=p(\alpha)$ is the set defined by $\mathrm{Re}p(z)>\alpha$
3. $p(1,0)=p(1)$ is the set defined by $|p(z)-1|<1$
4. $p(\alpha,0)=p_*(\alpha)$ is the set defined by $|p(z)-1|<\alpha$
5. $p(1,\frac{1}{M}-1)=p(M)$ is the set defined by $|p(z)-M|<M$
6. $p(\alpha,-\alpha)=p_{**}(\alpha)$ is the set defined by $\left|\frac{p(z)-1}{p(z)+1}\right|<\alpha$.

1.3 The class of Janowski's Starlike Functions

Let $S^*(A,B)$ be the class of functions $f(z)$, $f(0)=0$, $f'(0)=1$ regular in $\mathcal{D}$ and satisfying the condition

$$f(z)\in S^*(A,B) \quad \text{if and only if} \quad \frac{zf'(z)}{f(z)}\in p(A,B). \tag{1.3}$$

Special selections of A and B lead to familiar sets defined by the inequality under the conditions $M>\frac{1}{2}$, $0\le\alpha<1$, we have

1. $S^*(-1,1)=S^*$ is the class of starlike functions with respect to the origin
2. $S^*(1-2\alpha,-1)=S^*(\alpha)$ is the class of starlike functions of order α
3. $S^*p(1,0)=S^*(1)$ is the class defined by $|\frac{zf'(z)}{f(z)}-1|<1$
4. $S^*p(\alpha,0)=S^*_*(\alpha)$ is the class defined by $|\frac{zf'(z)}{f(z)}-1|<\alpha$, $0\le\alpha<1$
5. $S^*(1,\frac{1}{M}-1)=S^*(M)$ is the class defined by $|\frac{zf'(z)}{f(z)}-M|<M$, $M>\frac{1}{2}$
6. $S^*(\alpha,-\alpha)=S^*_{**}(\alpha)$ is the class defined by $\left|\frac{\frac{zf'(z)}{f(z)}-1}{\frac{zf'(z)}{f(z)}+1}\right|<\alpha$.

Functions f in S^* and $S^*(\alpha)$ respectively are called the Janowski starlike functions and Janowski starlike functions of order α [5]. Since the condition $\left|\frac{w(z)-1}{w(z)+1}\right|<\alpha$, $z\in\mathcal{D}$, $w(z)=\frac{zf'(z)}{f(z)}$ implies that $|w(z)-m|<M$ where $M=\frac{1+\alpha^2}{1-\alpha^2}$.
For this latter class $S^*(\alpha)$, Parvatham proved the following:

Theorem 1.1 *[1]. Let $\gamma\ge0$, $0<\alpha\le1$ and δ be given by*

$$\delta:=\alpha\left[\frac{2+\alpha+\gamma(1-\alpha)}{1+2\alpha+\gamma(1-\alpha)}\right]. \tag{1.4}$$

If $f \in S^*(\delta)$, then the function $\mathcal{F}(z)$ given by Bernardi's integral,

$$F = I_\gamma f(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} f(t)dt. \tag{1.5}$$

is in $S^*(\alpha)$.

Theorem 1.2 *[1]. Let $\gamma \geq 0$, $0 < \alpha \leq 1$ and δ be given by*

$$\delta := \alpha\left[\frac{2-\alpha+\gamma(1-\alpha)}{1+\gamma(1-\alpha)}\right]. \tag{1.6}$$

If $f \in R(\delta)$, then the function $\mathcal{F}(z)$ given by Bernardi's integral (1.5) is in $R(\alpha)$.

In this paper, we study the integral operator such that

$$\mathcal{F} = I_{\beta,\gamma} f(z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t) t^{\gamma-1} dt\right]^{1/\beta}, \tag{1.7}$$

where β, $\gamma \in \mathbb{C}$, with $\beta \neq 0$, $\beta > 0$, $\gamma \geq 0$, $Re(\beta+\gamma) > 0$ and $f \in S$. Bulboacă [4] and Bernardi [2] showed that the classes: K of convex functions, S^* of starlike functions were closed under the transform (1.7) and (1.5), respectively.

Furthermore, in this paper we determine δ so that whenever $f \in S^*(\alpha)$ we have $\mathcal{F} \in S^*(\delta)$, and also, we consider a similar problem for

$$R_\beta(\alpha) = \left|\frac{\frac{f'(z)}{(f(z))^{1-\beta}} - 1}{\frac{f'(z)}{(f(z))^{1-\beta}} + 1}\right| < \alpha, \qquad z \in \mathcal{D}. \tag{1.8}$$

Setting $p(z) = \frac{f'(z)}{(f(z))^{1-\beta}}$, we can rewrite the condition of (1.8) in the form of the condition given by item number 6 in sections 1.2 and 1.3, or equivalent to

$$Re\{p(z)\} > \frac{1+\alpha}{1-\alpha}.$$

Here $R_1(1)$ is the class of $f \in S$ such that f' belongs to the *Caratheodory* class $w(z)$ of functions and by Parvatham [1], $R_1(\alpha)$ is the class of $f \in S$ such that f' belongs to $R(\alpha)$.

Lemma 1.3 *[3] Suppose that the functions $w(z)$ is regular in $\mathcal{D}$ with $w(0) = 0$. Then if $|w(z)|$ attains its maximum value on a circle $|z| = r < 1$ at a point $z_o \in \mathcal{D}$, we have*

1. $z_o w'(z_o) = k z_o$
2. $Re\left\{1 + \frac{z_o w''(z_o)}{w'(z_o)}\right\} \geq k$ *where $k \geq 1$ is a real number.*

2 Main Results

Theorem 2.1 *Let $\mathcal{F}$ be given by (1.7) and $f \in S^*(\delta)$, where*

$$\delta = \frac{2\alpha(3\beta+\gamma-1)+(3\beta-1-2\gamma)\alpha^2+\beta-1}{2\alpha(3\beta-\gamma-1)+(\beta-1)\alpha^2+(3\beta-1+2\gamma)}.$$

Then $\mathcal{F} \in S^*(\alpha)$ for all $\beta,\ \gamma \in \mathbb{C}$, with $\beta \neq 0$, $\beta > 0$, $\gamma \geq 0$, $0 < \alpha \leq 1$.

Proof. First, we define a function $w(z)$ by

$$w(z) = \frac{1}{\alpha}\left\{\frac{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}-1}{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}+1}\right\}, \quad \text{for } 0<\alpha\leq 1. \tag{2.9}$$

and $w(z) \neq 1$ for $z \in \mathcal{D}$. Then $w(z)$ is analytic in $\mathcal{D}$ and $w(0) = 0$. It is suffices to show that $|w(z)| < 1$ in $\mathcal{D}$.

From (2.9) we have

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} = \frac{1+\alpha w(z)}{1-\alpha w(z)};$$

by taking the logarithmic derivative we obtain

$$1+\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} = \frac{2\alpha z w'(z)}{1-\alpha^2 w^2(z)} + \frac{1+\alpha w(z)}{1-\alpha w(z)}. \tag{2.10}$$

From (1.7) with simple differentiation, we have

$$(\beta+\gamma)f^{\beta}(z) = \mathcal{F}^{\beta}(z)\left[\beta\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}+\gamma\right].$$

By logarithmic differentiation yields

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\left\{\frac{\beta\left(\frac{1+z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right)-\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}}{\beta\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}+\gamma}+1\right\}\\ &= \frac{1+\alpha w(z)}{1-\alpha w(z)}\left\{\frac{\frac{2\beta\alpha z w'(z)}{1+\alpha w(z)}+(\beta-1)(1+\alpha w(z))}{\beta(1+\alpha w(z))+\gamma(1-\alpha w(z))}+1\right\},\end{aligned}$$

such that

$$\frac{zf'(z)}{f(z)} = \frac{2\beta\alpha z w'(z)+(\beta-1)(1+\alpha w(z))^2}{(1-\alpha w(z))\left[\beta(1+\alpha w(z))+\gamma(1-\alpha w(z))\right]}+\frac{1+\alpha w(z)}{1-\alpha w(z)}.$$

According to lemma 1.3, we assume that there exist a point $z_o \in \mathcal{D}$, and

$$z_o w'(z_o) = k z_o, \qquad k \geq 1.$$

Then we get

$$\left\{ \frac{\frac{z_o f'(z_o)}{f(z_o)} - 1}{\frac{z_o f'(z_o)}{f(z_o)} + 1} \right\} = \frac{\alpha w(z_o)[(2\beta k + 4\beta - 2 + 2\gamma) + (3\beta - 1 - 2\gamma)\alpha w(z_o)] + \beta - 1}{\alpha w(z_o)[(2\beta k + 4\beta - 2 - 2\gamma) + (\beta - 1)\alpha w(z_o)] + (3\beta - 1 + 2\gamma)}$$

and

$$\begin{aligned} \left| \frac{\frac{z_o f'(z_o)}{f(z_o)} - 1}{\frac{z_o f'(z_o)}{f(z_o)} + 1} \right| &= \frac{\left|\alpha e^{i\theta}[(2\beta k + 4\beta - 2 + 2\gamma) + (3\beta - 1 - 2\gamma)\alpha e^{i\theta}] + \beta - 1\right|}{\left|\alpha e^{i\theta}[(2\beta k + 4\beta - 2 - 2\gamma) + (\beta - 1)\alpha e^{i\theta}] + (3\beta - 1 + 2\gamma)\right|} \\ &= \sigma(\theta), \end{aligned} \tag{2.11}$$

where $\sigma(\theta)$

$$= \frac{\left\{\left[(\beta k+2\beta-1+\gamma)2\alpha\cos\theta+(3\beta-1-2\gamma)\alpha^2\cos 2\theta+(\beta-1)\right]^2+\left[(\beta k+2\beta-1+\gamma)2\alpha\sin\theta+(3\beta-1-2\gamma)\alpha^2\sin 2\theta\right]^2\right\}^{1/2}}{\left\{[(\beta k+2\beta-1-\gamma)2\alpha\cos\theta+(\beta-1)\alpha^2\cos 2\theta+(3\beta-1+2\gamma)]^2+[(\beta k+2\beta-1-\gamma)2\alpha\sin\theta+(\beta-1)\alpha^2\sin 2\theta]^2\right\}^{1/2}}.$$

Let $\varphi(t)$

$$= \frac{4\alpha^2(\beta k+2\beta-1+\gamma)^2+\alpha^4(3\beta-1-2\gamma)^2+(\beta-1)^2+2\alpha\left[\alpha(2t^2-1)(3\beta-1-2\gamma)(\beta-1)+(\beta k+2\beta-1+\gamma)[(3\beta-1-2\gamma)\alpha^2+(\beta-1)]4t\right]}{4\alpha^2(\beta k+2\beta-1-\gamma)^2+\alpha^4(\beta-1)^2+(3\beta-1+2\gamma)^2+2\alpha\left[\alpha(2t^2-1)(3\beta-1+2\gamma)(\beta-1)+(\beta k+2\beta-1-\gamma)[(3\beta-1+2\gamma)\alpha^2+(\beta-1)]4t\right]};$$

we can show that $\varphi(t)$ is a decreasing function of $t = \cos\theta$ in $[-1, 1]$ for $\gamma \geq 0$, $\beta \geq 0$.

Hence from (2.11), we obtain

$$\left\{ \frac{\frac{z_o f'(z_o)}{f(z_o)} - 1}{\frac{z_o f'(z_o)}{f(z_o)} + 1} \right\} \geq \frac{2\alpha(3\beta + \gamma - 1) + (3\beta - 1 - 2\gamma)\alpha^2 + \beta - 1}{2\alpha(3\beta - \gamma - 1) + (\beta - 1)\alpha^2 + (3\beta - 1 + 2\gamma)} = \delta$$

which contradicts the hypothesis that $f \in S^*(\delta)$. Hence we have

$$|w(z)| = \frac{1}{\alpha} \left| \frac{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1}{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + 1} \right| < 1,$$

or $\mathcal{F} \in S^*(\alpha)$ which completes the proof of the theorem. □

For $0 < \alpha \leq 1$, we have $\alpha < \delta$ in Theorem 2.1 and obtain the following corollaries:

Corollary 2.2 *If $f \in S^*(\delta)$ then $\mathcal{F} \in S^*(\delta)$.*

Corollary 2.3 *If $\delta = 1$ and $\beta = 1$, then α reduces to one for all $\beta \geq 0$ and $\gamma \geq 0$ which is a result of Bernardi [2].*

Remark 2.4 *If we take $\beta = 1$ in Theorem 2.1, we obtain the results (1.4) given by Parvatham [1].*

Theorem 2.5 *Let $\mathcal{F}$ be given by (1.7) and $f \in R_\beta(\eta)$, where*

$$\eta = \frac{2\alpha(1+\beta+\gamma) - (\beta+2\gamma+1)\alpha^2 - (\beta-1)}{2\alpha(1-(\beta+\gamma)) + (\beta-1)\alpha^2 + (\beta+2\gamma+1)}.$$

Then $\mathcal{F} \in R_\beta(\alpha)$ for all $\beta, \gamma \in \mathbb{C}$, with $\beta \neq 0$, $\beta > 0$, $\gamma \geq 0$ and $0 < \alpha \leq 1$.

Proof.

Let us define a function $w(z)$ by

$$w(z) = \frac{1}{\alpha}\left\{\frac{\frac{\mathcal{F}'(z)}{(\mathcal{F}(z))^{1-\beta}} - 1}{\frac{\mathcal{F}'(z)}{(\mathcal{F}(z))^{1-\beta}} + 1}\right\}, \quad \text{for } 0 < \alpha \leq 1. \tag{2.12}$$

and $w(z) \neq 1$ for $z \in \mathcal{D}$, such that $w(z)$ is analytic in $\mathcal{D}$ and $w(0) = 0$. It is suffices to show that $|w(z)| < 1$ in $\mathcal{D}$.

From (2.12) we have

$$\frac{\mathcal{F}'(z)}{(\mathcal{F}(z))^{1-\beta}} = \frac{1+\alpha w(z)}{1-\alpha w(z)};$$

and by taking the logarithmic derivative, we obtain

$$(\beta-1)\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} = \frac{2\alpha z w'(z)}{1-\alpha^2 w^2(z)}. \tag{2.13}$$

Differentiate (1.7), we obtain

$$\begin{aligned}\frac{f'(z)}{(f(z))^{1-\beta}} &= \frac{\mathcal{F}'(z)}{(\mathcal{F}(z))^{1-\beta}}\left\{\left(\beta\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + \gamma\right) + \left(1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} - \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)\right\}\frac{1}{\beta+\gamma} \\ &= \frac{\mathcal{F}'(z)}{(\mathcal{F}(z))^{1-\beta}}\left\{(\beta-1)\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} + (\gamma+1)\right\}\frac{1}{\beta+\gamma}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\frac{f'(z)}{(f(z))^{1-\beta}}-1}{\frac{f'(z)}{(f(z))^{1-\beta}}+1} &= \frac{\frac{\mathcal{F}'(z)}{(\mathcal{F}(z))^{1-\beta}}\left\{(\beta-1)\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}+\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}+(\gamma+1)\right\}\frac{1}{\beta+\gamma}-1}{\frac{\mathcal{F}'(z)}{(\mathcal{F}(z))^{1-\beta}}\left\{(\beta-1)\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}+\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}+(\gamma+1)\right\}\frac{1}{\beta+\gamma}+1}\\
&= \frac{\left(\frac{1+\alpha w(z)}{1-\alpha w(z)}\right)\left\{\frac{2\alpha z w'(z)}{1-\alpha^2 w^2(z)}+(\gamma+1)\right\}\frac{1}{\beta+\gamma}-1}{\left(\frac{1+\alpha w(z)}{1-\alpha w(z)}\right)\left\{\frac{2\alpha z w'(z)}{1-\alpha^2 w^2(z)}+(\gamma+1)\right\}\frac{1}{\beta+\gamma}+1}\\
&= \frac{2\alpha z w'(z)+2\alpha(\beta+\gamma)w(z)-(\beta+2\gamma+1)\alpha^2 w^2(z)-(\beta-1)}{2\alpha z w'(z)-2\alpha(\beta+\gamma)w(z)+(\beta-1)\alpha^2 w^2(z)+(\beta+2\gamma+1)}.\end{aligned}$$

According to Lemma 1.3, assume that there exist a point $z_o \in \mathcal{D}$ such that $\max\limits_{|z|\leq|z_o|} |w(z)| = |w(z_o)| = 1$, and

$$z_o w'(z_o) = k z_o, \qquad k \geq 1.$$

Then we obtain

$$\left\{\frac{\frac{f'(z_o)}{(f(z_o))^{1-\beta}}-1}{\frac{f'(z_o)}{(f(z_o))^{1-\beta}}+1}\right\} = \frac{2\alpha z w'(z_o)+2\alpha(\beta+\gamma)w(z_o)-(\beta+2\gamma+1)\alpha^2 w^2(z_o)-(\beta-1)}{2\alpha z w'(z_o)-2\alpha(\beta+\gamma)w(z_o)+(\beta-1)\alpha^2 w^2(z_o)+(\beta+2\gamma+1)}$$

and

$$\begin{aligned}\left|\frac{\frac{f'(z_o)}{(f(z_o))^{1-\beta}}-1}{\frac{f'(z_o)}{(f(z_o))^{1-\beta}}+1}\right| &= \frac{\left|2\alpha(k+\beta+\gamma)e^{i\theta}-(\beta+2\gamma+1)\alpha^2 e^{i2\theta}-(\beta-1)\right|}{\left|2\alpha(k-(\beta+\gamma))e^{i\theta}+(\beta-1)\alpha^2 e^{i2\theta}+(\beta+2\gamma+1)\right|}\\
&= \tau(\theta), \qquad (2.14)\end{aligned}$$

where $\tau(\theta)$

$$= \frac{\left\{\left[2\alpha(k+\beta+\gamma)\cos\theta-(\beta+2\gamma+1)\alpha^2\cos 2\theta-(\beta-1)\right]^2+\left[2\alpha(k+\beta+\gamma)\sin\theta-(\beta+2\gamma+1)\alpha^2\sin 2\theta\right]^2\right\}^{1/2}}{\left\{[2\alpha(k-(\beta+\gamma))\cos\theta+(\beta-1)\alpha^2\cos 2\theta+(\beta+2\gamma+1)]^2+[2\alpha(k-(\beta+\gamma))\sin\theta+(\beta-1)\alpha^2\sin 2\theta]^2\right\}^{1/2}}.$$

Let $\Psi(t)$

$$= \frac{4\alpha^2(k+\beta+\gamma)^2+\alpha^4(\beta+2\gamma+1)^2+(\beta-1)^2-4\alpha(k+\beta+\gamma)\left[\alpha^2(\beta+2\gamma+1)-(\beta-1)\right]t-2\alpha^2(\beta+2\gamma+1)(\beta-1)(2t^2-1)}{4\alpha^2(k-(\beta+\gamma))^2+\alpha^4(\beta-1)^2+(\beta+2\gamma+1)^2+4\alpha(k-(\beta+\gamma))\left[\alpha^2(\beta-1)+(\beta+2\gamma+1)\right]t+2\alpha^2(\beta+2\gamma+1)(\beta-1)(2t^2-1)};$$

we can show that $\Psi(t)$ is a decreasing function of $t = \cos\theta$ in $[-1, 1]$ for $\gamma \geq 0$, $\beta \geq 0$.

Hence from (2.14) we obtain

$$\left\{\frac{\frac{f'(z_o)}{(f(z_o))^{1-\beta}}-1}{\frac{f'(z_o)}{(f(z_o))^{1-\beta}}+1}\right\} \geq \frac{2\alpha(1+\beta+\gamma)-(\beta+2\gamma+1)\alpha^2-(\beta-1)}{2\alpha(1-(\beta+\gamma))+(\beta-1)\alpha^2+(\beta+2\gamma+1)} = \eta$$

which contradicts the hypothesis that $f \in R_\beta(\eta)$. Hence we have

$$|w(z)| = \frac{1}{\alpha}\left|\frac{\frac{f'(z)}{(f(z))^{1-\beta}}-1}{\frac{f'(z)}{(f(z))^{1-\beta}}+1}\right| < 1,$$

or $\mathcal{F} \in R_\beta(\alpha)$ which completes the proof of the theorem. □

For $0 < \alpha \leq 1$, we have $\alpha < \eta$ in Theorem 2.5 and obtain the following corollaries:

Corollary 2.6 *If $f \in R_\beta(\eta)$ then $\mathcal{F} \in R_\beta(\eta)$.*

Corollary 2.7 *If $\eta = 1$ and $\beta = 1$, then α reduces to one for all $\beta \geq 0$ and $\gamma \geq 0$ which is a result of Bernardi [2].*

Remark 2.8 *If we take $\beta = 1$ in Theorem 2.5, we obtain the results (1.6) given by Parvatham [1].*

Acknowledgement The work presented here was supported by SAGA: STGL-012-2006, Academy of Sciences, Malaysia.

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Received: April 19, 2008