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Solution of the Linear and Non-Linear Schrödinger Equations Using Homotopy Perturbation and Adomian Decomposition Methods

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Abstract

In this paper, homotopy perturbation method (HPM) and Adomian decomposition method (ADM) are employed to solve the linear and non-linear Schrödinger equations. To illustrate the capability and reliability of the methods, some examples are provided. The results obtained using HPM and ADM are compared with the results of variational iteration method (VIM) and then the ability of each method are also discussed.

Keywords: Schrödinger equation; Homotopy perturbation method; Adomian decomposition method; Variational iteration method

1 Introduction

Homotopy perturbation method was established in 1999 by He [1]. The method is a powerful and efficient technique to find the solutions of non-linear equations. The coupling of the perturbation and homotopy methods is called the homotopy perturbation method. This method can take the advantages of the conventional perturbation method while eliminating its restrictions. HPM has been applied by many authors [2-4] to solve many types of the linear and non-linear equations in science and engineering.

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2 The linear and non-linear Schrödinger equations

The linear Schrödinger equation can be expressed in the following form [5]:

$$u_t + iu_{xx} = 0, \tag{1}$$

$$u(x,0) = f(x).$$

There also exists another type of this equation which is called the non-linear Schrödinger and can be expressed as follows [5]:

$$iu_t + u_{xx} + \eta|u|^2 u = 0, \tag{2}$$

$$u(x,0) = f(x).$$

where η is a constant. These equations also occur in the non-linear optics, plasma physics, superconductivity and quantum mechanics.

3 Homotopy perturbation method (HPM)

To illustrate the basic idea of this method, we consider the following non-linear differential equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{3}$$

with the following boundary conditions:

$$B(u, \frac{\partial u}{\partial n}) = 0, \qquad r \in \Gamma, \tag{4}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A can be decomposed into a linear part and a non-linear one, designated as L and N respectively. Hence Eq.(3) can be written as the following form:

$$L(u) + N(u) - f(r) = 0.$$

Using homotopy technique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \longrightarrow R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{5}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq.(3) which satisfies the boundary conditions. Obviously, from Eq.(5) we will have:

$$H(v,0) = L(v) - L(u_0) = 0,$$

$$H(v,1) = A(v) - f(r) = 0.$$

By changing the value of p from zero to unity, $v(r,p)$ change from $u_0(r)$ to $u(r)$, in topology this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. Due to the fact that $p \in [0,1]$ can be considered as a small parameter, thus we consider the solution of Eq.(5) as a power series in p as the following:

$$v = v_0 + pv_1 + p^2 v_2 + \dots, \tag{6}$$

setting $p = 1$ results in the approximate solution for Eq.(3)

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots. \tag{7}$$

4 Adomian decomposition method (ADM)

According to the analysis of Adomian decomposition method, Eq.(2) can be written in the following operator form:

$$iLu + Ru + Nu = 0, \qquad u(x,0) = f(x), \tag{8}$$

where $L = \frac{\partial}{\partial t}$ and $R = \frac{\partial^2}{\partial x^2}$ are the linear operators and $Nu = \eta |u|^2 u$ is a non-linear term.

Adomian decomposition method assumes that the unknown function $u(x,t)$ can be expressed by an infinite series in the form:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{9}$$

and the non-linear term $Nu = \eta |u|^2 u = \eta u^2 \bar{u}$ can be decomposed into an infinite series of polynomials given by

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n), \tag{10}$$

where the components $u_n(x,t)$ are usually determined recurrently and $A_n(u_0, u_1, \dots, u_n)$ are called Adomian polynomials that are defined by

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n} N(\sum_{i=0}^{\infty} \lambda^i u_i)]_{\lambda=0}, \qquad n = 0, 1, 2, \dots. \tag{11}$$

The Adomian polynomials can be calculated for all forms of nonlinearity according to algorithms set by Adomian [6]. There are also the other different methods to calculate Adomian polynomials (for example, see [7] and the references therein).

The inverse operator L^{-1} is an integral operator given by $L^{-1}(\cdot) = \int_0^t (\cdot)dt$. Applying $-iL^{-1}$ to both sides of Eq.(8), we obtain:

$$u(x,t) = u(x,0) + iL^{-1}(Ru + Nu). \tag{12}$$

Substituting Eqs. (9) and (10) into (12) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + iL^{-1}[R(\sum_{n=0}^{\infty} u_n) + \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n)]. \tag{13}$$

From (13) we define:

$$u_0(x,t) = u(x,0), \tag{14}$$

$$u_{n+1}(x,t) = iL^{-1}(Ru_n + A_n) \qquad n = 0, 1, 2, \ldots. \tag{15}$$

5 Test examples

Example 1. Consider the following linear Schrödinger equation with the indicated initial condition:

$$u_t + iu_{xx} = 0, \tag{16}$$

$$u(x,0) = e^{(3ix)}.$$

HPM: Using HPM, we construct the following homotopy:

$$H(v,p) = (1-p)[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}] + p[\frac{\partial v}{\partial t} + i\frac{\partial^2 v}{\partial x^2}] = 0. \tag{17}$$

Substituting Eq.(6) into Eq.(17) and equating the terms with identical powers of p leads to:

$$\begin{aligned}
p^0 &: \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \qquad v_0(x,0) = e^{(3ix)},\\
p^1 &: \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} + i\frac{\partial^2 v_0}{\partial x^2} = 0, \qquad v_1(x,0) = 0,\\
p^2 &: \frac{\partial v_2}{\partial t} + i\frac{\partial^2 v_1}{\partial x^2} = 0, \qquad v_2(x,0) = 0,\\
&\vdots
\end{aligned} \tag{18}$$

Consider $u_0(x,t) = e^{(3ix)}$ as a first approximation for the solution that satisfies the initial condition. Solving the Eqs.(18) successively leads to:

$$v_0(x,t) = e^{(3ix)},$$

$$v_1(x,t) = (9it)e^{(3ix)},$$

$$v_2(x,t) = \frac{(9it)^2}{2!}e^{(3ix)},$$

$$\vdots$$

Hence, the solution of Eq.(16) when $p \longrightarrow 1$ will be as the following:

$$u(x,t) = e^{(3ix)} + (9it)e^{(3ix)} + \frac{(9it)^2}{2!}e^{(3ix)} + \ldots = e^{3i(x+3t)},$$

which is an exact solution and is the same as that reported in [5].

ADM: Using $u_0(x,t) = u(x,0)$ and $u_{n+1} = -i\int_0^t (\frac{\partial^2 u_n}{\partial x^2})dt \qquad n = 0,1,2,\ldots,$ we drive:

$$u_0(x,t) = e^{(3ix)},$$

$$u_1(x,t) = (9it)e^{(3ix)},$$

$$u_2(x,t) = \frac{(9it)^2}{2!}e^{(3ix)},$$

$$\vdots$$

Hence:

$$u(x,t) = e^{(3ix)} + (9it)e^{(3ix)} + \frac{(9it)^2}{2!}e^{(3ix)} + \ldots = e^{3i(x+3t)},$$

which is an exact solution and is the same as that obtained by VIM [5].

Example 2. Consider the following non-linear Schrödinger equation with the indicated initial condition:

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{19}$$

$$u(x,0) = e^{ix}.$$

HPM: Using HPM, we construct a homotopy in the following form:

$$H(v,p) = (1-p)[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}] + p[\frac{\partial v}{\partial t} - i(\frac{\partial^2 v}{\partial x^2} + 2v^2\bar{v})] = 0. \tag{20}$$

where $v^2\bar{v} = |v|^2 v$ and $\bar{v}$ is the conjugate of v.

Substituting Eq.(6) into Eq.(20) and equating the terms with identical powers of p, leads to:

$$\begin{aligned} p^0 &: \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \qquad v_0(x,0) = e^{ix}, \\ p^1 &: \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} - i(\frac{\partial^2 v_0}{\partial x^2} + 2v_0^2\bar{v}_0) = 0, \qquad v_1(x,0) = 0, \\ p^2 &: \frac{\partial v_2}{\partial t} - i(\frac{\partial^2 v_1}{\partial x^2} + 2(v_0^2\bar{v}_1 + 2v_1 v_0 \bar{v}_0)) = 0, \qquad v_2(x,0) = 0, \\ &\vdots \end{aligned} \tag{21}$$

Let's select $u_0(x,t) = e^{ix}$ as an initial approximation that satisfies the initial condition. From Eqs.(21), the following terms can be computed successively:

$$\begin{aligned} v_0(x,t) &= e^{ix}, \\ v_1(x,t) &= (it)e^{ix}, \\ v_2(x,t) &= \frac{(it)^2}{2!}e^{ix}, \\ &\vdots \end{aligned}$$

Therefore, the solution of Eq.(19) when $p \longrightarrow 1$ will be as follows:

$$u(x,t) = e^{ix} + (it)e^{ix} + \frac{(it)^2}{2!}e^{ix} + \ldots = e^{i(x+t)},$$

which is an exact solution and is the same as that reported in [5].

ADM: Applying Eqs.(14) and (15), we will drive:

$$\begin{aligned} u_0(x,t) &= e^{ix}, \\ u_1(x,t) &= (it)e^{ix}, \\ u_2(x,t) &= \frac{(it)^2}{2!}e^{ix}, \\ &\vdots \end{aligned}$$

Therefore:

$$u(x,t) = e^{ix} + (it)e^{ix} + \frac{(it)^2}{2!}e^{ix} + \ldots = e^{i(x+t)},$$

which is an exact solution and is the same as that obtained by VIM [5].

6 Conclusion

In this paper, homotopy perturbation and Adomian decomposition methods have been successfully applied to solve the linear and non-linear Schrödinger equations. It was shown that these methods are very efficient and powerful to get the exact solution. Comparison among HPM, ADM and VIM shows that although the results of these methods when applied to solve the Schrödinger equation are in good agreement, HPM can overcome the difficulties arising in calculation of Adomian's polynomials. Also this method does not require construction of correction functional using general lagrange multipliers, such as VIM and is much easier and more convenient than ADM and VIM. The computations associated with the examples provided here were performed by using Maple 10.

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