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On the Domination Number of Some Graphs

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Abstract

Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is a dominating set of graph G, if every vertex in $V - S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. It is well known that if $e \in E(G)$, then $\gamma(G-e)-1 \leq \gamma(G) \leq \gamma(G-e)$. In this paper, as an application of this inequality, we obtain the domination number of some certain graphs.

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1 Introduction

Let $G = (V, E)$ be a graph of order $|V| = n$. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. For a detailed treatment of this parameter, the reader is referred to [3].

In this paper, as an application of the following theorem, we obtain the domination number of some certain graphs.

As usual we use $\lceil x \rceil$ for the smallest integer greater than or equal to x.

Theorem 1. ([4]) *If* $e \in E(G)$*, then* $\gamma(G - e) - 1 \leq \gamma(G) \leq \gamma(G - e)$. □

In the next section we recall the definition of graphs $G(m)$ with specific construction, and as example of these types of graph we obtain the domination number of some certain trees. In Section 3, we obtain the domination number of other graph in the form $G(m)$ denoted by $C_n(m)$.

2 Domination number of $T_n(m)$

We first recall the graphs $G(m)$ defined in [1]. Let P_{m+1} be a path with vertices labeled by $y_0, y_1, \ldots, y_m$, for $m \geq 0$ and let v_0 be a specific vertex of a graph G. Denote by $G_{v_0}(m)$ (or simply $G(m)$) a graph obtained from G by identifying the vertex v_0 of G with an end vertex y_0 of P_{m+1} (see Figure 1).

For example, if G is a path P_2, then $G(m) = P_2(m)$ is a path P_{m+2}. As another example of this graph we consider tree denoted by $T_n(m)$ as show in the Figure 1, which is special case of $P_n(m)$. In other words $T_n(m)$ is a tree, such that $T_n(0) = P_n$, and for $m \geq 1$, $T_n(m) = (A, B, E)$ where $A \cup B$ is its vertex set, $A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_m\}$, and the edge set $E = \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{b_i b_{i+1} : 1 \leq i \leq m-1\} \cup \{a_{n-1} b_1\}$.

In this section, as an application of Theorem 1, we obtain the domination number of $T_n(m)$. Note that we can obtain the domination number of $T_n(m)$ by combinatorial ways which are easier than the following way. We use Theorem 1 and present computational way to obtain the domination numbers of some trees. We need the following lemmas in the proof of Theorem 2:

Lemma 1.([2], p.371) $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. □

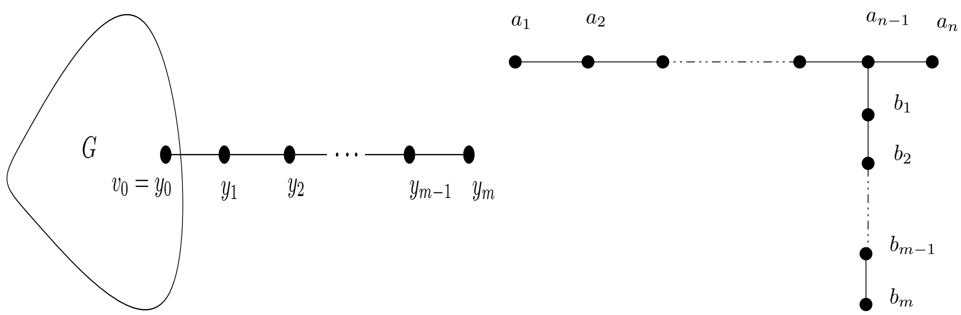

Figure 1: The graphs $G(m)$ and $T_n(m)$, respectively.

The following lemma follows from our observation.

Lemma 2. *For every $n, m \geq 3$, $\gamma(T_n(m)) = \gamma(T_{m+2}(n-2))$.* □

The following theorem give us the domination number of $T_n(m)$.

Theorem 2. *For every $n \geq 3$ and $m \geq 0$, $\gamma(T_n(m)) = \lceil \frac{n}{3} \rceil + \lceil \frac{m-1}{3} \rceil$.*

Proof. By induction on m. If $m = 0$, then $\gamma(T_n(0)) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$. Now suppose that the theorem is true for all numbers less than or equal $m-1$, and we prove it for m. By applying Theorem 1 for $e = a_{n-1}b_1$, we have the following inequalities:

$$\lceil \frac{n}{3} \rceil + \lceil \frac{m}{3} \rceil - 1 \leq \gamma(T_n(m)) \leq \lceil \frac{n}{3} \rceil + \lceil \frac{m}{3} \rceil. \tag{1}$$

Similarly by applying theorem 1 for $e = b_2b_3$ and $e = b_{m-3}b_{m-2}$, and by induction hypothesis we have the following inequalities:

$$\lceil \frac{n}{3} \rceil + \lceil \frac{m-2}{3} \rceil \leq \gamma(T_n(m)) \leq \lceil \frac{n}{3} \rceil + \lceil \frac{m-2}{3} \rceil + 1. \tag{2}$$

$$\lceil \frac{n}{3} \rceil + \lceil \frac{m-4}{3} \rceil \leq \gamma(T_n(m)) \leq \lceil \frac{n}{3} \rceil + \lceil \frac{m-4}{3} \rceil + 1. \tag{3}$$

Now if $m = 3k$ for some $k \in N$, then by (1) and (2) we have

$$\gamma(T_n(m)) = \lceil \frac{n}{3} \rceil + k = \lceil \frac{n}{3} \rceil + \lceil \frac{m-1}{3} \rceil.$$

If $m = 3k+1$ for some $k \in N$, then by (1) and (3) we have

$$\gamma(T_n(m)) = \lceil \frac{n}{3} \rceil + k = \lceil \frac{n}{3} \rceil + \lceil \frac{m-1}{3} \rceil.$$

Now if $m = 3k + 2$ for some $k \in N$, we shall consider the following cases:
(i) $n = 3k'$ for some $k' \in N$. By applying Theorem 1 for $e = a_{n-1}a_n$, we have the following inequalities:

$$\lceil \frac{n+m-1}{3} \rceil \leq \gamma(T_n(m)) \leq \lceil \frac{n+m-1}{3} \rceil + 1. \quad (4)$$

Now, by (1) and (4) we have

$$\gamma(T_{3k'}(3k+2)) = k + k' + 1 = \lceil \frac{n}{3} \rceil + \lceil \frac{m-1}{3} \rceil.$$

(ii) $n = 3k' + 2$ for some $k' \in N$. By Theorem 1 for $e = a_1a_2$ we have

$$\lceil \frac{n-1}{3} \rceil + \lceil \frac{m-1}{3} \rceil \leq \gamma(T_n(m)) \leq \lceil \frac{n-1}{3} \rceil + \lceil \frac{m-1}{3} \rceil + 1. \quad (5)$$

By (1) and (5) we have

$$\gamma(T_{3k'+2}(3k+2)) = k + k' + 2 = \lceil \frac{n}{3} \rceil + \lceil \frac{m-1}{3} \rceil.$$

(iii) $n = 3k' + 1$ for some $k' \in N$. We shall prove that $\gamma(T_{3k'+1}(3k+2)) = k + k' + 2$. We do it by induction on k. If $k = 0$, then by Lemma 2, $\gamma(T_{3k'+1}(2)) = \gamma(T_4(3k'-1)) = 2 + k'$. So the result is true for $k = 0$. Now suppose that the result is true for all number less than k, and we prove it for k. By Lemma 2 and induction hypothesis, we have the following equalities for $k' - 1 < k$:

$$\begin{aligned} \gamma(T_{3k'+1}(3k+2)) &= \gamma(T_{3k+4}(3k'-1)) = \gamma(T_{3(k+1)+1}(3(k'-1)+2)) \\ &= k' - 1 + k + 1 + 2 \\ &= k' + k + 2. \end{aligned}$$

If $k' - 1 > k$, then for some $t > 0$, $k' = k + 1 + t$. Again by Lemma 2,

$$\begin{aligned} \gamma(T_{3k'+1}(3k+2)) &= \gamma(T_{3k+3t+4}(3k+2)) = \gamma(T_{3k+4}(3(k+t)+2)) \\ &= k + t + k + 1 + 2 = k' + k + 2 \end{aligned}$$

Finally, for $k' - 1 = k$, we have

$$\begin{aligned} \gamma(T_{3k'+1}(3k+2)) &= \gamma(T_{3k+4}(3k+2)) = \gamma(T_{3(k+1)+1}(3k+2) \\ &= 2k + 3 = k + k' + 2. \end{aligned}$$

Therefore in all cases $\gamma(T_n(m)) = \lceil \frac{n}{3} \rceil + \lceil \frac{m-1}{3} \rceil$. □

Note that the tree $T_n(m)$ is obtained by gluing an end vertex of a path P_m to the vertex a_{n-1} of P_n. We now consider another type of related tree $T_{n,m}(k)$ (or special case of $(T_n(m))(k)$), which is obtained from $T_n(m)$ by gluing an end vertex of a path P_k to the vertex a_2 of P_n. See Figure 2. In other words, let $T_{n,m}(k) = (A, B, C, E)$, $k, n \geq 0$, where $A \cup B \cup C$ is its vertex set, $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\}$, $C = \{c_1, \ldots, c_k\}$, and the edge set $E = \{a_ia_{i+1} : 1 \leq i \leq n-1\} \cup \{b_ib_{i+1} : 1 \leq i \leq m-1\} \cup \{c_ic_{i+1} : 1 \leq i \leq k-1\} \cup \{a_{n-1}b_1, a_2c_1\}$.

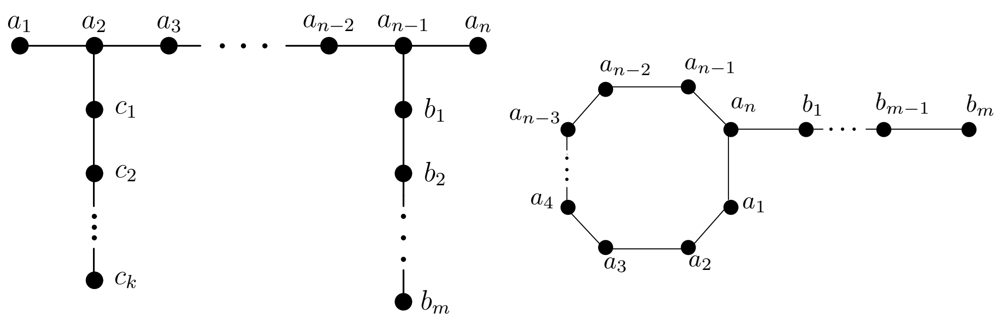

Figure 2: The graphs $T_{n,m}(k)$ and $C_n(m)$, respectively.

Theorem 3. *For every $n \geq 3$, and $m, k \geq 0$,*

$$\gamma(T_{n,m}(k)) = \lceil \frac{n}{3} \rceil + \lceil \frac{m-1}{3} \rceil + \lceil \frac{k-1}{3} \rceil.$$

Proof. Similar to the proof of Theorem 2. □

3 Domination number of $C_n(m)$

In this section as another example of $G(m)$, we consider $C_n(m)$ as shown in Figure 2.

Theorem 4. *for every $n \geq 3$ and $m \geq 0$, $\gamma(C_n(m)) = \lceil \frac{n}{3} \rceil + \lceil \frac{m-1}{3} \rceil$.*

Proof. We shall prove that

$$\gamma(C_n(m)) = \begin{cases} \lceil \frac{n}{3} \rceil + k & \text{if } m = 3k, 3k+1, \text{ for some } k \geq 0; \\ \lceil \frac{n}{3} \rceil + k + 1 & \text{if } m = 3k+2, \text{ for some } k \geq 0. \end{cases}$$

By Theorem 1 for $C_n(m)$ and $e = a_n b_1$ we have the following inequalities:

$$\lceil\frac{n}{3}\rceil + \lceil\frac{m}{3}\rceil - 1 \leq \gamma(C_n(m)) \leq \lceil\frac{n}{3}\rceil + \lceil\frac{m}{3}\rceil. \tag{6}$$

By Theorem 1 for $e = a_1 a_2$ and Theorem 2 we have:

$$\lceil\frac{n}{3}\rceil + \lceil\frac{m-1}{3}\rceil - 1 \leq \gamma(C_n(m)) \leq \lceil\frac{n}{3}\rceil + \lceil\frac{m-1}{3}\rceil. \tag{7}$$

By $(6),(7)$ for $m = 3k+1$ we have $\gamma(C_n(3k+1)) = \lceil\frac{n}{3}\rceil + k$. By Theorem 1 for $e = b_5 b_6$ we have

$$\lceil\frac{n}{3}\rceil + \lceil\frac{m-5}{3}\rceil + 1 \leq \gamma(C_n(m)) \leq \lceil\frac{n}{3}\rceil + \lceil\frac{m-5}{3}\rceil + 2. \tag{8}$$

By (6) and (8) for $m = 3k$, we have $\gamma(C_n(3k)) = \lceil\frac{n}{3}\rceil + k$. With similar arguments, one can prove that $\gamma(C_n(3k+2)) = \lceil\frac{n}{3}\rceil + k + 1$, and the proof is complete. □

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