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P-Spaces and the Prime Spectrum of Commutative Semirings

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Abstract

By a *semiring* we understand a commutative semiring with nonzero identity. The notions of ω-*absolutely* (*semiprime-*) *irreducible* ideals in a semiring R are introduced and we prove that the prime spectrum Spec(R) of R is a P-space if and only if every prime ideal of R is ω-absolutely semiprime-irreducible. Also, Spec(R) is an Artin space totally-ordered by inclusion if and only if every (semiprime) ideal of R is ω-absolutely semiprime-irreducible if and only if the family Spec(R) is well-ordered by inclusion. Further, we characterize when every ideal of R is ω-absolutely irreducible as well as when every (semiprime) prime ideal of R is ω-absolutely irreducible.

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1 Introduction

In this work we continue the study of the prime spectrum of a semiring ([4]). In section 2 we include some preliminaries. In section 3, we introduce the notions of *ω-absolutely (semiprime-) irreducible* ideals in a semiring R, and we prove that a prime ideal of R is a P-point of the prime spectrum Spec(R) of R if and only if it is ω-absolutely semiprime-irreducible. So, Spec(R) is a P-space if and only if every prime ideal of R is ω-absolutely semiprime-irreducible (Theorems 3.2-3.3). Also, Spec(R) is an Artin space totally-ordered by inclusion if and only if every (semiprime) ideal of R is ω-absolutely semiprime-irreducible if and only if the family Spec(R) is well-ordered by inclusion (Theorem 3.7). In section 4 we characterize the ω-absolutely irreducible ideals of R (Theorem 4.1) and we prove that every ideal of R is ω-absolutely irreducible if and only if R satisfies the descending chain condition on principal ideals and these ideals are totally-ordered by inclusion (Theorem 4.2). Finally, every prime (resp. semiprime) ideal of R is ω-absolutely irreducible if and only if Spec(R) is a P-space (resp. Artin space) and the prime radical function η preserves countable intersections (Theorems 4.3 and 4.4).

2 Preliminaries

On the following, $\mathbb{N} := \{0, 1, 2, \dots\}$ denotes the set of natural numbers, ω is the cardinal of $\mathbb{N}$ and for every nonempty family $\mathcal{F}$ of subsets of a set X, we denote by $\mathcal{F}^{\cap}$ the family formed by all the intersections of sets in $\mathcal{F}$. By a *space* we understand a topological space and (X, τ) denotes a space.

Recall that a *semiring* (commutative with nonzero identity) is an algebra $(R, +, \cdot, 0, 1)$, where R is a set with $0, 1 \in R$, and $+$ and $\cdot$ are binary operations on R called *sum* and *multiplication*, respectively, which satisfy the following:

(S_1) $(R, +, 0)$ and $(R, \cdot, 1)$ are commutative monoids with $1 \neq 0$.

(S_2) $a \cdot (b + c) = a \cdot b + a \cdot c$ for every $a, b, c \in R$.

(S_3) $a \cdot 0 = 0$ for every $a \in R$.

We denote a semiring $(R, +, \cdot, 0, 1)$ by R, and the multiplication $a \cdot b$ by ab. Also, Id(R) (resp. Spec(R)) denotes the family of ideals (resp. prime ideals) of R, and for every ideal I of R, $\eta(I)$ is the *prime radical* of I, this is, the intersection of the prime ideals of R containing I. It is well-known that $\eta(I) = \{a \in R : \exists\, n \in \mathbb{N}, a^n \in I\}$ and we say I is *semiprime* if $I = \eta(I)$. So, the improper ideal R is semiprime (the empty intersection of prime ideals), and for every $x \in R$, we set $Rx := \{rx : r \in R\}$ and $\eta(x) := \eta(Rx)$. See [2]-[3] for more details about the semiring theory and their applications.

3 P-points in the prime spectrum of semirings

We recall the construction of the prime spectrum of a semiring R. For every ideal I of R, we denote by $(I)_0$ the set of prime ideals of R containing I and by $D_0(I) := \mathrm{Spec}(R)\backslash(I)_0$. Also, $(x)_0 := (Rx)_0$ and $D_0(x) := D_0(Rx)$ for every $x \in R$. Then, it is easy see that:

(*i*) $(1)_0 = \emptyset$ and $(0)_0 = \mathrm{Spec}(R)$.

(*ii*) $(\sum_{j\in S} I_j)_0 = \bigcap_{j\in S}(I_j)_0$ for every family $\{I_j\}_{j\in S}$ of ideals of R.

(*iii*) $(I)_0 \cup (J)_0 = (IJ)_0 = (I \cap J)_0$ for every pair I, J of ideals of R.

So, the collection $\{(I)_0 : I \in \mathrm{Id}(R)\}$ satisfies the axioms of closed sets for a topology t_Z on $\mathrm{Spec}(R)$, called the *Zariski topology*, and the space $(\mathrm{Spec}(R), t_Z)$ is the *prime spectrum* of R. Recall that, in a space (X, τ), a *G_δ-subset* of X is a countable intersection of τ-open subsets of X and a point $x \in X$ is a *P-point* if every G_δ-subset G of X containing x is a *neighborhood* of x in (X, τ), this is, there exists $U \in \tau$ such that $x \in U \subseteq G$ ([1], [9]). We denote by $\tau(x)$ is the set of τ-open subsets of X containing x. For every subset Y of X, we denote by $\overline{Y}^{\tau}$ the τ-closure of Y. So, $\overline{\mathcal{F}}^{t_Z} = (\bigcap \mathcal{F})_0$ for every subset $\mathcal{F}$ of $\mathrm{Spec}(R)$.

Lemma 3.1 *A point x of a space (X, τ) is a P-point if and only if whenever x is a closure point of subset F of X and G is a G_δ-subset of X containing x we have $G \cap F \neq \emptyset$.*

Proof. The necessary condition is clear. Suppose the sufficiency condition and let G be a G_δ-subset of X containing x. If G is not a neighborhood of x then every τ-open subset of X containing x intersects $F = X\backslash G$ and so, $x \in \overline{F}^{\tau}$ and $G \cap F \neq \emptyset$ which is a contradiction. □

Let I be an ideal of a semiring R and $\mathcal{F}$ a nonempty family of ideals of R. Following [4], we say that I is *ω-absolutely $\mathcal{F}$-irreducible* if for every sequence $\{I_n\}$ in $\mathcal{F}$ such that $\bigcap_{n\in\mathbb{N}} I_n \subseteq I$, there exists $n \in \mathbb{N}$ such that $I_n \subseteq I$. In the case $\mathcal{F} = \mathrm{Spec}(R)^{\cap}$ we say that I is *ω-absolutely semiprime-irreducible*, and if $\mathcal{F} = \mathrm{Id}(R)$ then we say that I is *ω-absolutely irreducible*.

Theorem 3.2 *Let R be a semiring and P a prime ideal of R. Then, the following conditions are equivalent:*

(*i*) *P is a P-point of* $\mathrm{Spec}(R)$.

(*ii*) *P is an ω-absolutely semiprime-irreducible ideal.*

(*iii*) *If P is a t_Z-closure point of subset $\mathcal{F}$ of* $\mathrm{Spec}(R)$ *and $\mathcal{G}$ is a G_δ-subset of* $\mathrm{Spec}(R)$ *containing P then $\mathcal{G} \cap \mathcal{F} \neq \emptyset$.*

Proof. $(i) \Leftrightarrow (iii)$ by Lemma 3.1. We see $(iii) \Rightarrow (ii)$. Suppose (iii) and let $\mathcal{F} = \{I_n\}_{n\in\mathbb{N}}$ be a sequence of proper semipime ideals of R such that $\bigcap_{n\in\mathbb{N}} I_n \subseteq P$. Also, let $\mathcal{F}^* = \bigcup_{n\in\mathbb{N}}(I_n)_0$. Then, $\bigcap \mathcal{F} = \bigcap \mathcal{F}^*$ and P is a closure point of $\mathcal{F}^*$. Now, if every $I_n \nsubseteq P$ then $\mathcal{G} = \bigcap_{n\in\mathbb{N}} D_0(I_n)$ is a G_δ-subset of $\operatorname{Spec}(R)$ containing P and by hypothesis, $\mathcal{G}$ intersects $\mathcal{F}^*$ which is a contradiction. We see $(ii) \Rightarrow (i)$. Suppose (ii) and let $\mathcal{G}$ be a G_δ-subset of $\operatorname{Spec}(R)$ such that $P \in \mathcal{G}$. Then, there exists a sequence $\{I_n\}_{n\in\mathbb{N}}$ of proper semiprime ideals of R such that $\mathcal{G} = \bigcap_{n\in\mathbb{N}} D_0(I_n)$ and so, every $I_n \nsubseteq P$ and by hypothesis, there exists $a \in (\bigcap_{n\in\mathbb{N}} I_n)\backslash P$. It follows $P \in D_0(a) \subseteq \mathcal{G}$. □

Theorem 3.3 *Let R be a semiring. Then, are equivalent:*

(i) $\operatorname{Spec}(R)$ *is a P-space.*

(ii) *Every prime ideal of R is ω-absolutely semiprime-irreducible.*

(iii) *If P is a prime ideal R in the t_Z-closure of a subset $\mathcal{F}$ of* $\operatorname{Spec}(R)$ *and $\mathcal{G}$ is a G_δ-subset of* $\operatorname{Spec}(R)$ *containing P then $\mathcal{G} \cap \mathcal{F} \neq \emptyset$.*

Proof. It is an immediate consequence of Theorem 3.2. □

A set is *countable* if its cardinal is at most ω, this is, it is finite or ω. A pre-ordered set $(X, \leq)$ is *countably well-ordered* if every nonempty countable subset of X have a first element respect to $\leq$. Also, $(X, \leq)$ *satisfies the descending chain condition* (in short, *dcc*) if for every descending chain $x_0 \geq x_1 \geq x_2 \geq \cdots$ in X, there exists $m \geq 1$ such that $x_m = x_n$ for every $n \geq m$.

Lemma 3.4 *Let $(X, \leq)$ be a pre-ordered set. Then, $(X, \leq)$ is (countably) well-ordered if and only if $(X, \leq)$ is totally-ordered and satisfies dcc.*

Proof. The necesary condition is clear. Suppose the sufficiency condition and that S is a nonempty subset of X without first element respect to $\leq$. Then, there exists $x_1 \in S$ and since x_1 is not a first element of S, there exists $x_2 \in S$ such that $x_1 > x_2$, this is, $x_2 \leq x_1$ and $x_1 \neq x_2$. Analoguously, there exists $x_3 \in S$ such that $x_2 > x_3$ and so, we can construct a strictly descending chain $x_1 > x_2 > x_3 > \cdots$ in S (and hence, in X) which is a contradiction. □

We see that the condition "every (semiprime) ideal of a semiring R is ω-absolutely semiprime-irreducible" characterize the artinianity of $\operatorname{Spec}(R)$ when it is totally-ordered by inclusion. Recall that (X, τ) is *Artin space* if the poset $(\tau, \subseteq)$ satisfies dcc.

Proposition 3.5 *Every Artin space is a P-space. Further, if (X, τ) is an Artin* T_0*-space then the poset $(X, \leq_\tau)$ satisfies dcc.*

Proof. Let (X, τ) be an Artin space and $G = \bigcap_{n\in\mathbb{N}} U_n$ a G_δ-subset of X where every $U_n \in \tau$. For every $n \in \mathbb{N}$, we set $V_n = \bigcap_{j=0}^{n} U_j$. Then, $\{V_n\}$ is a descending chain in $(\tau, \subseteq)$ and so, there exists $m \in \mathbb{N}$ such that $V_m = V_n$ for every $n \geq m$. Hence, $G = \bigcap_{n\in\mathbb{N}} V_n = V_m \in \tau$ and (X, τ) is a P-space. On the other hand, suppose (X, τ) is an Artin T_0-space and that $\{x_n\}_{n=1}^{\infty}$ is a strictly descending chain in $(X, \leq_\tau)$, this is, $x_1 \gneqq_\tau x_2 \gneqq_\tau x_3 \gneqq_\tau \cdots$. Then, $x_1 \in \overline{x_2}^\tau$ and $x_2 \notin \overline{x_1}^\tau$ (by T_0). Let $V_1 \in \tau(x_1)$. Then, $x_2 \in V_1$ and there exists $U_2 \in \tau(x_2)$ such that $x_1 \notin U_2$. Let $V_2 = V_1 \cap U_2$. Then, $x_2 \in V_2$ and $V_1 \supsetneqq V_2$ (since $x_1 \in V_1 \backslash V_2$). Analogously, $x_2 \in \overline{x_3}^\tau$ and $x_3 \notin \overline{x_2}^\tau$. So, $x_3 \in V_2$ and there exist $U_3 \in \tau(x_3)$ such that $x_2 \notin U_3$. Let $V_3 = V_2 \cap U_2$. Then, $V_2 \supsetneqq V_3$ and continuing of this way, we can construct a strictly descending sequence $\{V_n\}$ in τ which is a contradiction. □

An infinite discrete space is a P-space which is not Artin. Also, in [5] is proved that if (X, τ) is a strongly irreducible space such that $(X, \leq_\tau)$ satisfies dcc then (X, τ) is an Artin space. Recall that a space is *strongly irreducible* if the intersection of any family of nonempty open subsets is nonempty ([8]).

Lemma 3.6 *Let R be a semiring. Then, are equivalent:*

(i) *The poset* $(\mathrm{Spec}(R)^\cap, \subseteq)$ *is (countably) well-ordered.*

(ii) *Every semiprime ideal of R is ω-absolutely semiprime-irreducible.*

(iii) *The poset* $(\mathrm{Spec}(R), \subseteq)$ *is (countably) well-ordered.*

Further, in such a case, every proper semiprime ideal of R is prime.

Proof. We see $(i) \Leftrightarrow (ii)$. Suppose (i) and that $\bigcap_{n\in\mathbb{N}} H_n \subseteq I$ where I and every H_n is a semiprime ideal of R. Then, exists the first element H_m of the family $\{H_n\}$ and so, $H_m \subseteq I$. Conversely, suppose (ii). By Lemma 3.4, we need prove only that $(\mathrm{Spec}(R)^\cap, \subseteq)$ is countably well-ordered. Let $\mathcal{F} = \{H_n\}$ be a sequence of semiprime ideals of R. Then, $I = \bigcap_{n\in\mathbb{N}} H_n$ is a semiprime ideal of R and by hypothesis, there exists $m \in \mathbb{N}$ such that $I = H_m$ and so, it is the fierst element of $\mathcal{F}$. Finally, $(ii) \Rightarrow (iii)$ is as in $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ as in $(i) \Rightarrow (ii)$. The last part is clear. □

Theorem 3.7 *Let R be a semiring. Then, are equivalent:*

(i) $\mathrm{Spec}(R)$ *is an Artin space totally-ordered by inclusion.*

(ii) *Every semiprime ideal of R is ω-absolutely semiprime-irreducible.*

(iii) *The poset* $(\mathrm{Spec}(R)^\cap, \subseteq)$ *is (countably) well-ordered.*

(iv) *Every ideal of R is ω-absolutely semiprime-irreducible.*

(*v*) *The poset* $(\mathrm{Spec}(R)^{\cap}, \subseteq)$ *satisfies dcc and* $(\mathrm{Spec}(R), \subseteq)$ *is totally-ordered.*

(*vi*) *The poset* $(\mathrm{Spec}(R), \subseteq)$ *is (countably) well-ordered.*

(*vii*) *The poset* $(\mathrm{Spec}(R), \subseteq)$ *is totally-ordered and satisfies dcc.*

(*viii*) *For every nonempty (countable) subset* S *of* R*, there exists* $a \in S$ *such that* $a \in \eta(b)$ *for every* $b \in S$*.*

Proof. It is clear that $(iii) \Rightarrow (viii)$ and by Lemmas 3.4 and 3.6, $(i) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$. We see $(viii) \Rightarrow (iii)$. Suppose $(viii)$ and that $\mathcal{F}$ is a nonempty countable family of semiprime ideals of R without first element and let $I_1 \in \mathcal{F}$. Then, there exist $I_2 \in \mathcal{F}$ such that $I_1 \nsubseteq I_2$ and $a_1 \in I_1 \backslash I_2$. Analogously, there exist $I_3 \in \mathcal{F}$ and $a_2 \in I_2 \backslash I_3$. So, we can construct two sequences $\{I_n\}$ in $\mathrm{Spec}(R)^{\cap}$ and $\{a_n\}$ in R such that $a_n \in I_n \backslash I_{n+1}$ for every $n \in \mathbb{N}$. But then, by hypothesis, there exists $m \in \mathbb{N}$ such that $a_m \in \eta(a_n)$ for every $n \in \mathbb{N}$ and so, $a_m \in \eta(a_{m+1}) \subseteq I_{m+1}$ which is a contradiction. □

The argument in $(vii) \Rightarrow (iii)$ of Theorem 3.7 is a version of the proof of Theorem 2.3 in [4]. Also, if we set $a \leq_\eta b$ in R if $a \in \eta(b)$ then $\leq_\eta$ is a pre-order on R such that $0 \leq_\eta a \leq_\eta 1$ for every $a \in R$. Further, $\leq_\eta$ is antisymmetric (and hence, a partial order on R) if and only if R is multiplicatively idempotent and 1 is the unique invertible element of R (Theorem 2.1 in [6]). Further, is clear that the condition (vii) in Theorem 3.7 is equivalent to $(R, \leq_\eta)$ is a well-ordered pre-ordered set.

4 ω-Absolutely irreducible ideals

We set $a \leq_R b$ in R if $a \in Rb$. Then, $\leq_R$ is a pre-order on R such that $0 \leq_R a \leq_R 1$ for every $a \in R$. This pre-order is considered in [7]. We denote by $\mathrm{Id}_1(R)$ the family of principal ideals of R, and for every subset S of R, we set $I_S := \bigcap_{a \in S} Ra$. Also, extending the notion of i-system in [2], we say that a nonempty subset A of R is an *ω-system* if for every sequence S in A, we have $I_S \cap A \neq \emptyset$. So, a nonempty subset of R is an ω-system (resp. i-system) if and only if $(R, \leq_R)$ is well-ordered (resp. totally-ordered) set.

Theorem 4.1 *Equivalent conditions for an ideal* I *of a semiring* R*:*

(*i*) I *is* ω*-absolutely irreducible.*

(*ii*) I *is* ω*-absolutely* $\mathrm{Id}_1(R)$*-irreducible.*

(*iii*) $R \backslash I$ *is an* ω*-system.*

Proof. It is clear that $(i) \Rightarrow (ii)$. We see $(ii) \Rightarrow (iii)$. Suppose (ii) and let S be a sequence in $A = R\backslash I$ such that $I_S \cap A = \emptyset$. Then, $I_S \subseteq I$ and there exists $x \in S$ such that $Rx \subseteq I$ which is a contradiction (since $x \in A$). We see $(iii) \Rightarrow (i)$. Suppose (iii) and let $\{I_n\}$ be a sequence of proper ideals of R such that $\bigcap_{n\in\mathbb{N}} I_n \subseteq I$. If every $I_n \nsubseteq I$ there exists $x_n \in I_n\backslash I$ for every index $n \in \mathbb{N}$ and so, $S = \{x_n\}_{n\in\mathbb{N}}$ is a sequence in $R\backslash I$ such that $I_S \cap (R\backslash I) = \emptyset$, which is a contradiction (since $I_S \subseteq \bigcap_{n\in\mathbb{N}} I_n \subseteq I$). □

Theorem 4.2 *Equivalent conditions for a semiring R:*

(i) *Every ideal of R is ω-absolutely irreducible.*

(ii) *For every countable subset S of R, we have $I_S \cap S \neq \emptyset$.*

(iii) $(R, \leq_R)$ *is a (countably) well-ordered pre-ordered set.*

(iv) *The poset* $(\mathrm{Id}_1(R), \subseteq)$ *is (countably) well-ordered.*

(v) *The poset* $(\mathrm{Id}_1(R), \subseteq)$ *is totally-ordered and satisfies dcc.*

(vi) *The poset* $(\mathrm{Id}(R), \subseteq)$ *is (countably) well-ordered.*

(vii) *The poset* $(\mathrm{Id}(R), \subseteq)$ *is totally-ordered and satisfies dcc.*

Further, in such a case, $\mathrm{Spec}(R)$ *is an Artin space totally-ordered by inclusion.*

Proof. It is clear that $(iii) \Leftrightarrow (iv)$, and the equivalences $(i) - (iii)$ and $(iv) - (vii)$ follows from Theorem 4.1 and Lemma 3.4. The last part is clear. □

Note that the notions of prime ideal and ω-absolutely irreducible ideal are independent (consider the ring of integers). We consider the *prime radical function* $\eta : \mathrm{Id}(R) \to \mathrm{Id}(R)$ defined by: $I \mapsto \eta(I)$ for every ideal I of R. So, $\eta(R) = R$ and, in general, η is monotone and preserves finite intersections.

Theorem 4.3 *Every prime ideal of a semiring R is ω-absolutely irreducible if and only if* $\mathrm{Spec}(R)$ *is a P-space and η preserves countable intersections.*

Proof. Suppose that every prime ideal of R is ω-absolutely irreducible. Then, $\mathrm{Spec}(R)$ is a P-space (Theorem 3.3). Now, let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of ideals of R. Is clear that $\eta(\bigcap_{n\in\mathbb{N}} I_n) \subseteq \bigcap_{n\in\mathbb{N}} \eta(I_n)$ and if P is a prime ideal of R containing $\bigcap_{n\in\mathbb{N}} I_n$ then by hypothesis, there exists $m \in \mathbb{N}$ such that $I_m \subseteq P$ and so, $\eta(I_m) \subseteq P$ and we have the requiered equality. Conversely, suppose $\mathrm{Spec}(R)$ is a P-space and η preserves countable intersections. Let P be a prime ideal of R and $\{I_n\}$ a sequence of ideals of R such that $\bigcap_{n\in\mathbb{N}} I_n \subseteq P$. Then, $\eta(\bigcap_{n\in\mathbb{N}} I_n) = \bigcap_{n\in\mathbb{N}} \eta(I_n) \subseteq P$ and there exists $m \in \mathbb{N}$ such that $\eta(I_m) \subseteq P$ (Theorem 3.3). Hence, P is ω-absolutely irreducible. □

Theorem 4.4 *Every semiprime ideal of a semiring R is ω-absolutely irreducible if and only if* $\mathrm{Spec}(R)$ *is an Artin space and η preserves countable intersections.*

Proof. The necessary condition it follows of Theorems 3.7 and 4.3. Conversely, suppose $\mathrm{Spec}(R)$ is an artinian space and η preserves countable intersections. Let I be a semiprime ideal of R and $\{H_n\}$ a sequence of ideals of R such that $H = \bigcap_{n\in\mathbb{N}} H_n \subseteq I$. Then, $\eta(H) = \bigcap_{n\in\mathbb{N}} \eta(H_n) \subseteq I$ and by Theorem 3.7, there exists $m \in \mathbb{N}$ such that $\eta(H_m) \subseteq I$. □

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