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Analytic Approach to Solve Specific Linear and Nonlinear Difference Equations

L. Hassani Oskouei and N. Aliev

Azarbaijan University of Tarbiat Moallem, Tabriz, Iran
lh_oskouei@azaruniv.edu

Abstract. Discrete mathematics encounters two kinds of problems, the problems those are discrete naturally and those that one can alternate them with discrete models. In this paper, we consider specific linear and nonlinear difference equation and solve them analytically. Solving the associated Cauchy initial value and boundary value problems are carried out a well.

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1. Introduction

In numerical solving of an Ordinary Differential Equation (ODE), the concept of (continuous) derivative is replaced by its estimates such as differences and change an ODE to a difference equation. If an ODE is linear, its corresponding difference equation is a linear one. However, a nonlinear ODE might be converted to either a linear or nonlinear difference equation. There are many attempts to solve linear and nonlinear difference equations analytically [1, 2].

In this paper, specific linear and nonlinear difference equations with variational coefficients are considered. For these kind of deferential equations we find closed form of solutions that help us in solving corresponding Initial Value Problem (IVP) and Boundary Value Problems (BVP). The problem we are studying in this paper is a discrete one. The main idea in finding a closed form of solution is induction.

The paper organized as follows. Section 2 is devoted to solve some specific second order nonlinear difference equations. In Section 3, some second order linear difference equation with variational coefficients are considered and the closed form of their solution is obtained. Third order nonlinear difference equations are considered in Section 4. Some kinds of unsolved difference equations are presented in final section.

2. Nonlinear second order difference equation

Let us consider the following form of nonlinear discrete equation:

$$y_{i+2} = \left(h_i + \frac{y_{i+1}}{y_i}\right) y_{i+1}, \qquad i \geq 0, \tag{2.1}$$

where $\{h_i\}_{i=0}^{\infty}$ is a known real-valued sequence and $\{y_i\}_{i=0}^{\infty}$ is an unknown real-valued one. Substituting $i = 0$ and $i = 1$ in (2.1) leads to

$$y_2 = h_0 y_1 + \frac{y_1^2}{y_0}, \tag{2.2}$$

and

$$y_3 = \frac{y_1^3}{y_0^2} + (2h_0 + h_1)\frac{y_1^2}{y_0} + (h_0^2 + h_1 h_0) y_1, \tag{2.3}$$

respectively. One may concerned to the closed form of the solution of (2.1), where y_0 and y_1 are arbitrary constant real numbers. The following theorem talks about this fact.

Theorem 2.1. *For the equation (2.1), if the sequence $\{h_i\}_{i=0}^{\infty}$ is known, then*

$$y_n = y_1 \prod_{k=0}^{n-2} \left(\frac{y_1}{y_0} + \sum_{j=0}^{k} h_j \right), \qquad n \geq 2, \tag{2.4}$$

is its solution for all $n \geq 2$, where y_0 and y_1 are arbitrary constant real numbers.

Proof: *It is easy to verify that by letting for $n = 2, 3$ in (2.4), relations (2.2) and (2.3) are derived, respectively. We prove that if (2.4) holds for $n \leq k$, then it holds for $n = k + 1$ itself as well. To do this, considering $i = k - 1$ in (2.1) leads to*

$$y_{k+1} = \frac{y_k^2}{y_{k-1}} + h_{k-1} y_k. \tag{2.5}$$

Substituting y_k and y_{k-1} from the hypothesis of induction in (2.5) implies

$$\begin{aligned} y_{k+1} &= \frac{\left[y_1\prod_{s=0}^{k-2}\left(\frac{y_1}{y_0}+\sum_{j=0}^{s}h_j\right)\right]^2}{y_1\prod_{s=0}^{k-3}\left(\frac{y_1}{y_0}+\sum_{j=0}^{s}h_j\right)} + h_{k-1}y_1\prod_{s=0}^{k-2}\left(\frac{y_1}{y_0}+\sum_{j=0}^{s}h_j\right) \\ &= \left[y_1\prod_{s=0}^{k-2}\left(\frac{y_1}{y_0}+\sum_{j=0}^{s}h_j\right)\right]\left(\frac{y_1}{y_0}+\sum_{j=0}^{k-2}h_j\right) + h_{k-1}y_1\prod_{s=0}^{k-2}\left(\frac{y_1}{y_0}+\sum_{j=0}^{s}h_j\right) \\ &= \left[y_1\prod_{s=0}^{k-2}\left(\frac{y_1}{y_0}+\sum_{j=0}^{s}h_j\right)\right]\left(\frac{y_1}{y_0}+\sum_{j=0}^{k-2}h_j + h_{k-1}\right) \\ &= y_1\prod_{s=0}^{k-1}\left(\frac{y_1}{y_0}+\sum_{j=0}^{s}h_j\right), \end{aligned}$$

that completes the proof. □

The following observations are straightforward.

Remark 2.2. Let $y_0 = a$ and $y_1 = b$ be considered, then the unique solution of (2.1) if can be derived directly from (2.4). This is the closed form of the solution of the associated IVP. However, considering $y_0 = a$ and $y_m = b$ as boundary conditions, he associated BVP (2.1) has solution if and only if

$$y_1\prod_{k=0}^{m-2}\left(\frac{y_1}{a}+\sum_{j=0}^{k}h_j\right) = b, \tag{2.6}$$

has real solution for y_1.

Remark 2.3. If $\{h_i\}_{i=0}^{\infty}$ is a known complex sequence, then the associated IVP and BVP have solution but for the associated BVP, it might be not unique. However, equation (2.1) as a BVP with $y_1 = ay_0$ and $y_m = b$ has unique solution. Because, for $\frac{y_1}{y_0} = a$ we have

$$ay_0\prod_{k=0}^{m-2}\left(a+\sum_{j=0}^{k}h_j\right) = b,$$

with the condition $\prod_{k=0}^{m-2}\left(a+\sum_{j=0}^{k}h_j\right) \neq 0$ leads to

$$y_0 = \frac{b}{a}\prod_{k=0}^{m-2}\left(a+\sum_{j=0}^{k}h_j\right)^{-1} \text{ and } y_1 = b\prod_{k=0}^{m-2}\left(a+\sum_{j=0}^{k}h_j\right).$$

The uniqueness of the solution arises from (2.4). In special case, if $h_i = 0$ for $i \geq 0$, $y_0 = 1$ and $y_m = b > 0$, then equation (2.6) reduces to $y_1^m = b$. Thus, y_1

is the m-th root of b as

$$y_{1p} = \sqrt[m]{b}(\cos\frac{2\pi p}{m} + i\sin\frac{2\pi p}{m}), \qquad p = 1, \ldots, m.$$

Using these values lead to $y_{np} = y_{1p}^n$ for all $n \geq 0$.

3. Linear variational coefficient equation

Let us consider the linear variational coefficient equation

$$y_{i+2} = (g_i + 1)y_{i+1} - g_i y_i, \qquad i \geq 0, \tag{3.1}$$

where $\{g_i\}_{i=0}^{\infty}$ is a real-valued known sequence and $\{y_i\}_{i=0}^{\infty}$ is an unknown one. To find a closed form of solution of this problem, we prove the following theorem.

Theorem 3.1. *Let $\{g_i\}_{i=0}^{\infty}$ be a known real-valued sequence in (3.1), then the general solution of this equation is*

$$y_n = \left[1 + \sum_{k=0}^{n-2}\left(\prod_{j=0}^{k} g_j\right)\right] y_1 - \left[\sum_{k=0}^{n-2}\left(\prod_{j=0}^{k} g_j\right)\right] y_0, \qquad n \geq 2, \tag{3.2}$$

where y_0 and y_1 are arbitrary real constants.

Proof: *The proof is based on induction. For $i = 0, 1$ we have*

$$\begin{aligned} y_2 &= (g_0 + 1)y_1 - g_0 y_0, \\ y_3 &= (g_1 g_0 + g_0 + 1)y_1 - (g_1 g_0 + g_0)y_0, \end{aligned}$$

respectively. If (3.2) holds for $n \leq k$, one can easily prove that it holds for $n = k + 1$ as well. □

Remark 3.2. Analogous to Remark 2.2, one can conclude that the associated IVP to (3.1) accompanying initial conditions $y_0 = a$ and $y_1 = b$ has unique solution. Furthermore, for $y_0 = a$ and $y_m = b$ as boundary conditions, the associated BVP has solution if and only if

$$1 + \sum_{k=0}^{m-2}(\prod_{j=0}^{k} g_j) \neq 0. \tag{3.3}$$

In this case, its unique solution is

(3.4)

$$y_n = \left[1 + \sum_{k=0}^{n-2}\left(\prod_{j=0}^{k} g_j\right)\right] \frac{b + \left[\sum_{k=0}^{m-2}\left(\prod_{j=0}^{k} g_j\right)\right] a}{1 + \sum_{k=0}^{m-2}\left(\prod_{j=0}^{k} g_j\right)} - \left[\sum_{k=0}^{n-2}\left(\prod_{j=0}^{k} g_j\right)\right] a,$$

for all $n \geq 2$. On the other hand, if condition (3.3) does not hold, the associated BVP has no general solution. Meanwhile if (3.3) does not hold but

$$b + \left[\sum_{k=0}^{m-2} \left(\prod_{j=0}^{k} g_j \right) \right] a = 0,$$

holds, the associated BVP has solution but not unique. In fact this solution is obtained from (3.2) by considering $y_0 = a$ and y_1 as an arbitrary constant. However, with the condition $y_1 = y_0$ this problem has the unique solution constant $y_n = y_1$, $n \geq 2$.

4. Nonlinear third order difference equation

Let us consider the following variational coefficient linear difference equation

$$y_{i+3} = (h_i + 2)y_{i+2} - (2h_i + 1)y_{i+1} + h_i y_i, \tag{4.1}$$

for $i \geq 0$, where h_i, $i = 0, 1, \ldots$ are known real numbers. The following theorem refers to the general solution of this problem.

Theorem 4.1. *Let h_i, $i = 0, 1, \ldots$ be given real numbers. Then the general solution of (4.1), is*

$$y_n = y_2 + (n-2)y_1' + y_0'' \sum_{\eta=1}^{n-2} (n-1-\eta) \prod_{\xi=0}^{\eta-1} h_\xi, \tag{4.2}$$

for all $n \geq 3$, where y_i' and y_i'' denote the first and second order discrete derivative of y at $i = 0, 1$, and y_0, y_1 and y_2 are arbitrary real constants.
Proof: *For $i = 0$, one can reduce equation (4.1) to*

$$y_3 = (h_0 + 2)y_2 - (2h_0 + 1)y_1 + h_0 y_0, \tag{4.3}$$

and for $i = 1$ to

$$y_4 = (h_1 + 2)y_3 - (2h_1 + 1)y_2 + h_1 y_1. \tag{4.4}$$

Substituting (4.3) in (4.4) leads to

$$y_4 = (h_1 h_0 + 2h_0 + 3)y_2 - (2h_1 h_0 + 4h_0 + 2)y_1 + (h_1 h_0 + 2h_0)y_0. \tag{4.5}$$

It is easy to verify that equation (4.2), replacing n with 3 and 4, leads to (4.3) and (4.5), respectively. Let (4.2) holds for $n \leq k$. It is enough to show that it holds for $n = k + 1$ as well. Replacing $i = k - 2$ in (4.1) gives:

$$y_{k+1} = (h_{k-2} + 2)y_k - (2h_{k-2} + 1)y_{k-1} + h_{k-2} y_{k-2}.$$

Using the assumption of induction, y_k, y_{k-1} and y_{k-2} can be obtained and replaced from (4.2), one can verify the validity of (4.2) for $n = k + 1$. □

Remark 4.2. For the initial conditions $y_0 = a$ and $y_1 = b$ and $y_2 = c$ in (4.1), we have

$$y_2 = c,\ y_1' = c - b,\ y_0'' = c - 2b + a,$$

and the solution of IVP is easily concluded from (4.2). On the other hand, with boundary conditions $y_0 = a$, $y_1 = b$ and $y_m = c$ are given, the solution of the associated BVP can be obtained form (4.2) as

$$y_n = a\sum_{\eta=1}^{n-2}(n-1-\eta)\prod_{\xi=0}^{\eta-1}h_\xi - b\left[(n-2) + 2\sum_{\eta=1}^{n-2}(n-1-\eta)\prod_{\xi=0}^{\eta-1}h_\xi\right].$$

We can simplify this as:

$$y_n = +y_2\left[(n-1) + \sum_{\eta=1}^{n-2}(n-1-\eta)\prod_{\xi=0}^{\eta-1}h_\xi\right]. \tag{4.6}$$

where

$$\begin{aligned} c = {} & a\sum_{\eta=1}^{m-2}(m-1-\eta)\prod_{\xi=0}^{\eta-1}h_\xi - b\left[(m-2) + 2\sum_{\eta=1}^{m-2}(m-1-\eta)\prod_{\xi=0}^{\eta-1}h_\xi\right] \\ & + y_2\left[(m-1) + \sum_{\eta=1}^{m-2}(m-1-\eta)\prod_{\xi=0}^{\eta-1}h_\xi\right]. \end{aligned}$$

If

$$(m-1) + \sum_{\eta=1}^{m-2}(m-1-\eta)\prod_{\xi=0}^{\eta-1}h_\xi \neq 0, \tag{4.7}$$

then

$$y_2 = \frac{c - a\sum_{\eta=1}^{m-2}(m-1-\eta)\prod_{\xi=0}^{\eta-1}h_\xi + b\left[(m-2) + 2\sum_{\eta=1}^{m-2}(m-1-\eta)\prod_{\xi=0}^{\eta-1}h_\xi\right]}{(m-1) + \sum_{\eta=1}^{m-2}(m-1-\eta)\prod_{\xi=0}^{\eta-1}h_\xi}.$$

Using the value of y_2, one can easily obtain the solution of the associated BVP. The details are skipped.

In the sequel, we investigated another third order problem. Let us consider the following nonlinear difference equation:

$$y_{i+3} = g_i\frac{y_{i+2}^2}{y_{i+1}} + \frac{y_{i+2}^3}{y_{i+1}^3}y_i, \qquad i \geq 0. \tag{4.8}$$

where g_i is a real known number for $i \geq 0$. The following theorem talks about the general solution of (4.8).

Theorem 4.3. *Let g_i be given real constants in (4.8) for all $i \geq 0$. The general solution of problem (4.8) is*

$$y_n = y_2 \prod_{i=2}^{n-1} \left[\frac{y_2}{y_1} \prod_{j=1}^{i-1} \left(\frac{y_2 y_0}{y_1^2} + \sum_{k=0}^{j-1} g_k \right) \right], \quad n \geq 3, \tag{4.9}$$

where y_0, y_1 and y_2 are real known constants.

Proof: *For $i = 0$ we have*

$$y_3 = g_0 \frac{y_2^2}{y_1} + \frac{y_2^3}{y_1^3} y_0. \tag{4.10}$$

Moreover, for $i = 1$, it holds

$$y_4 = \frac{y_2^5}{y_1^6} y_0^2 (g_1 + 3g_0) + \frac{y_2^4}{y_1^4} y_0 (2g_0 g_1 + 3g_0^2) + \frac{y_2^3}{y_1^2} (g_1 g_0^2 + g_0^3) + \frac{y_2^6}{y_1^8} y_0^3. \tag{4.11}$$

Using mathematical induction, one can easily conclude the general solution of (4.8) as the theorem states.

Remark 4.4. For the initial conditions $y_0 = a,\ y_1 = b, y_2 = c$, the solution of the associated IVP (4.8) be obtained from (4.8). On the other hand, with the boundary conditions $y_1 = a y_0,\ y_2 = b y_1,\ y_m = c$, with the two first conditions, we have

$$y_n = y_2 \prod_{i=2}^{n-1} \left[b \prod_{j=1}^{i-1} \left(\frac{b}{a} + \sum_{k=0}^{j-1} g_k \right) \right], \quad n \geq 3.$$

and with accompanying the final condition to find y_2, it is necessary to have

$$\prod_{i=2}^{m-1} b \left[\prod_{j=1}^{i-1} \left(\frac{b}{a} + \sum_{k=0}^{j-1} g_k \right) \right] \neq 0, \tag{4.12}$$

and consequently,

$$y_n = c \frac{\prod_{i=2}^{n-1} \left[b \prod_{j=1}^{i-1} \left(\frac{b}{a} + \sum_{k=0}^{j-1} g_k \right) \right]}{\prod_{i=2}^{m-1} b \left[\prod_{j=1}^{i-1} \left(\frac{b}{a} + \sum_{k=0}^{j-1} g_k \right) \right]} = c \prod_{i=n}^{m-1} \left[b \prod_{j=1}^{i-1} \left(\frac{b}{a} + \sum_{k=0}^{j-1} g_k \right) \right]^{-1},$$

for $n \geq 3$. Finally, if $g_i = 0$ for all $i \geq 0$, then the solution of (3-7) is

$$y_n = \frac{y_2^{\frac{n(n-1)}{2}} y_0^{\frac{(n-2)(n-1)}{2}}}{y_1^{n(n-2)}}, \qquad n \geq 3.$$

5. Conclusion

In this paper, we solved some specific linear and nonlinear difference equation of order 2 and 3. There are some other specific linear and nonlinear difference equation. Finding their general solution are challenging.

1. Let us consider solving the following problem $y_{i+2} = (h_i + a_i y_i) y_{i+1} - (a_i - \frac{1}{y_i}) y_{i+1}^2$, where h_i is known for $i \geq 0$. Finding the general solution of its associated IVP is the goal.
2. Consider solving $y_{i=2} = h_i y_{i+1} + \frac{1-b_i}{y_i} y_{i+1}^2$, for $0 < i < m$, where h_i and b_i are given values. Finding the general solution of this problem and its associate BVP is another open problem.
3. The third open problem is finding the general solution of $y_{i+2} + (a_i y_i + g_i) y_i = [1 + 2a_i y_i + g_i - a_i y_{i+1}] y_{i+1}$, and its associated BVP where a_i, b_i, g_i are known for $0 < i < m$.
4. The next open problem is finding the general solution $y_{i+2} = (1 + g_i + b_i - b_i \frac{y_{i+1}}{y_i}) y_{i+1} - g_i y_i$, for $0 < i$ and its associated IVP.
5. The final open problem is finding the general solution of

$$y_{i+2} = \frac{a_i y_{i+1}^2 + (1 - a_i + f_i) y_i y_{i+1} - f_i y_i^2}{[(1 + a_i) y_{i+1} - a_i y_i] y_i} y_{i+1},$$

   where a_i, f_i are known for $0 < i$.

There are some other unanswered questions on these problems. Some of them are:

1. For the given problem (4.1), what is the fundamental solution, i.e. the general solution including singularity. Moreover, what are the necessary conditions for the BVP if we have this fundamental solution.
2. For the given problem (4.1) with non-local boundary conditions, what is the Green's sequence and what is its solution.
3. In problem (4.1), letting $h_i = \lambda \in \mathbb{C}$, with the nonlocal homogenous boundary conditions, what is the eigenvalue and eigne-sequence.
4. For the given problem (4.8), with nonlocal boundary value, what is the general solution.
5. What are necessary conditions for problem (4.8).
6. What-if when the boundary conditions are nonlinear in problem (4.8).

We are thinking of using additive and multiplicative discrete derivatives to solve this kind of problems.

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