Completely Generalized Set-Valued Strongly Nonlinear Mixed Variational-Like Inequalities in Hilbert Spaces

Jae Ug Jeong

Department of Mathematics, Dongeui University
Pusan 614-714, South Korea
jujeong@deu.ac.kr

Abstract. In this paper we will extend the auxiliary principle technique to study the completely generalized strongly nonlinear mixed variational-like inequality problem for set-valued mappings without compact values. We prove first the existence of a solution of the related auxiliary problem. Then, by using this existence result, we construct the iterative algorithm for solving that problem. And we show both the existence of a solution of the original problem and convergence of iterative sequences generated by the algorithm. The results in this paper extend and improve the corresponding results of [5, 8, 11].

Mathematics Subject Classification: 49A29, 49J40, 49D37

Keywords: Mixed variational-like inequality; auxiliary principle technique; iterative algorithm; set-valued mapping; Hilbert space

1. Introduction

Let $H$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $CB(H)$ be the family of all nonempty bounded closed subsets of $H$. Let $N, \eta : H \times H \rightarrow H$ be single-valued mappings and $T, A : H \rightarrow CB(H)$ be set-valued mappings.

In 2005, L. C. Zeng et al. [11] studied the following generalized set-valued strongly nonlinear mixed variational-like inequality: find $u \in H$, $w \in T(u)$, $y \in A(u)$ such that

$$\langle N(w, y), \eta(v, u) \rangle + b(u, v) - b(u, u) \geq 0, \quad \forall v \in H,$$

where $b(\cdot, \cdot) : H \times H \rightarrow \mathbb{R}$ is a real-valued function.
We further generalize this variational-like inequality as follows; find $u \in H$, $w \in T(u)$, $y \in A(u)$ such that

$$
\langle N(w, y), \eta(v, u) \rangle + b(u, v) - b(u, u) + a(u, v - u) \geq 0, \quad \forall v \in H,
$$

where $b(\cdot, \cdot) : H \times H \to \mathbb{R}$ is a real-valued function and $a : H \times H \to \mathbb{R}$ is a continuous function which is linear in both arguments. The variational-like inequality (1.2) is called the completely generalized set-valued strongly nonlinear mixed variational-like inequality.

In this paper we will extend the auxiliary principle technique to study the completely generalized strongly nonlinear mixed variational-like inequality problem (1.2) for set-valued mappings without compact values. We prove first the existence of a solution of the auxiliary problem for problem (1.2). Then, by using this existence result, we construct the iterative algorithm for solving problem (1.2). And we show both the existence of a solution of problem (1.2) and the convergence of iterative sequences generated by the algorithm. Our theorems improve and generalize some previous results in [5,8,11].

2. Preliminaries

Let $\mathbb{R} = (-\infty, +\infty)$, $H$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $CB(H)$ be the family of all nonempty bounded closed subsets of $H$. Let $a : H \times H \to \mathbb{R}$ be a continuous function which is linear in both arguments and there exist constants $\alpha > 0$, $\beta > 0$ satisfying

$$
a(u, u) \geq \alpha \| u \|^2, \quad \forall u \in H,
$$

$$
a(u, v) \leq \beta \| u \| \| v \|, \quad \forall u, v \in H.
$$

Let the functional $b : H \times H \to \mathbb{R}$ satisfy the following properties:

(i) $b(\cdot, \cdot)$ is linear in the first argument;

(ii) $b(\cdot, \cdot)$ is bounded, that is, there exists a constant $\gamma > 0$ such that

$$
b(u, v) \leq \gamma \| u \| \| v \|, \quad \forall u, v \in H;
$$

(iii) $b(u, v) - b(u, w) \leq b(u, v - w), \quad \forall u, v \in H$;

(iv) $b(\cdot, \cdot)$ is convex in the second argument.

**Remark 2.1.** In view of the properties (ii) and (iii), we know that

$$
\| b(u, v) - b(u, w) \| \leq \gamma \| u \| \| v - w \|, \quad \forall u, v, w \in H.
$$

This implies that $b(u, v)$ is continuous with respect to the second argument $v$. Note $b(\cdot, \cdot)$ is not necessarily differentiable.

Let $N, \eta : H \times H \to H$ be single-valued mappings and $T, A : H \to CB(H)$ be set-valued mappings. Now, we consider the following completely generalized...
set-valued strongly nonlinear mixed variational-like inequality problem: find $u \in H$, $w \in T(u)$ $y \in A(u)$ such that
\[ (2.3) \quad \langle N(w, y), \eta(v, u) \rangle + b(u, v) - b(u, u) + a(u, v - u) \geq 0, \quad \forall v \in H, \]
where $a$ and $b$ satisfy (2.1)-(2.2) and (i)-(iv), respectively.

**Definition 2.1.** Let $N : H \times H \to H$ be a nonlinear mapping and $T : H \to CB(H)$ be a set-valued mapping.

(i) $T$ is said to be strongly monotone with respect to the first argument of $N$ if there exists a constant $\alpha > 0$ such that
\[ \langle N(w_1, \cdot) - N(w_2, \cdot), u_1 - u_2 \rangle \geq \alpha \|u_1 - u_2\|^2, \]
\[ \forall u_1, u_2 \in H, w_1 \in T(u_1), w_2 \in T(u_2); \]

(ii) $N$ is said to be Lipschitz continuous with respect to the first argument if there exists a constant $\zeta > 0$ such that
\[ \|N(u_1, \cdot) - N(u_2, \cdot)\| \leq \zeta \|u_1 - u_2\|, \quad \forall u_1, u_2 \in H; \]

Similarly, we can define the Lipschitz continuity of a mapping $N(\cdot, \cdot)$ with respect to the second argument.

**Definition 2.2.** A mapping $\eta : H \times H \to H$ is said to be:

(i) strongly monotone if there exists a constant $\sigma > 0$ such that
\[ \langle \eta(u, v), u - v \rangle \geq \sigma \|u - v\|^2, \quad \forall u, v \in H; \]

(ii) Lipschitz continuous if there exists a constant $\delta > 0$ such that
\[ \|\eta(u, v)\| \leq \delta \|u - v\|, \quad \forall u, v \in H. \]

**Definition 2.3.** A set-valued mapping $V : H \to CB(H)$ is said to be Lipschitz continuous if there exists a constant $\nu > 0$ such that
\[ D(V(u), V(v)) \leq \nu \|u - v\|, \quad \forall u, v \in H, \]
where $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$.

Now we need the following assumption and Lemma which will be used in the proofs for the main results in the next sections.

**Assumption 2.1.** The mappings $N, \eta : H \times H \to H$ satisfy the following conditions:

(i) for all $w, y \in H$, there exists a constant $\tau > 0$ such that
\[ \|N(w, y)\| \leq \tau (\|w\| + \|y\|); \]

(ii) $\eta(v, u) = -\eta(u, v)$, $\forall u, v \in H$;

(iii) $\eta(\cdot, \cdot)$ is affine in the second argument;

(iv) for each fixed $u \in H$, $v \mapsto \eta(u, v)$ is continuous from the weak topology to the weak topology.
Lemma 2.1([10]). Let $X$ be a nonempty closed convex subset of a Hausdorff linear topological space $E$ and let $\phi, \psi : X \times X \to \mathbb{R}$ be mappings satisfying the following conditions:

1. $\psi(x, y) \leq \phi(x, y), \quad \forall x, y \in X,$
2. $\psi(x, x) \geq 0, \quad \forall x \in X$;
3. for each $x \in X$, $\phi(x, y)$ is upper semicontinuous with respect to $y$;
4. for each $y \in X$, the set $\{x \in X : \psi(x, y) < 0\}$ is convex;
5. there exists a nonempty compact set $K \subset X$ and $x_0 \in K$ such that $\psi(x_0, y) < 0$ for every $y \in X - K$.

Then there exists $\bar{y} \in K$ such that

$$\phi(x, \bar{y}) \geq 0, \quad \forall x \in X.$$

3. Auxiliary problem and algorithm

In this section, we extend the auxiliary principle technique due to Glowinski-Lions-Tremolières [4] to study the existence and uniqueness of solution for the completely generalized set-valued strongly nonlinear mixed variational-like inequality problem (2.3).

For given $x \in H$, $w \in T(u)$ and $y \in A(u)$, we consider the following auxiliary problem for the completely generalized set-valued strongly nonlinear variational-like inequality problem (2.3): find $z \in H$ such that

$$\langle z - u, v - z \rangle + \langle \rho N(w, y), \eta(v, z) \rangle + \rho [b(u, v) - b(u, z)] + \rho a(z, v - z) \geq 0, \quad \forall v \in H,$$

where $\rho > 0$ is a constant.

Theorem 3.1. Let the mapping $\eta : H \times H \to H$ be Lipschitz continuous with constant $\delta > 0$. Let $a : H \times H \to \mathbb{R}$ be a continuous function which is linear in both arguments and satisfy (2.1)-(2.2). Assume that $b : H \times H \to \mathbb{R}$ satisfies the properties (i)-(iv). If Assumption 2.1 holds, then the auxiliary problem (3.1) has a solution.

Proof. Define the mappings $\phi, \psi : H \times H \to \mathbb{R}$ by

$$\phi(v, z) = \langle v - u, v - z \rangle + \langle \rho N(w, y), \eta(v, z) \rangle + \rho [b(u, v) - b(u, z)] + \rho a(z, v - z),$$

$$\psi(v, z) = \langle z - u, v - z \rangle + \langle \rho N(w, y), \eta(v, z) \rangle + \rho [b(u, v) - b(u, z)] + \rho a(z, v - z),$$

where $u$ is fixed in $H$.

We will prove that the mappings $\phi$ and $\psi$ satisfy all the conditions of Lemma 2.1.
Indeed, we have
\[
\phi(v, z) = \langle v - u, v - z \rangle + \langle \rho N(w, y), \eta(v, z) \rangle + \rho [b(u, v) - b(u, z)] \\
+ \rho a(z, v - z)
\]
\[
= \langle v - z, v - z \rangle + \langle z - u, v - z \rangle + \langle \rho N(w, y), \eta(v, z) \rangle \\
+ \rho [b(u, v) - b(u, z)] + \rho a(z, v - z)
\]
\[
= \| v - z \|^2 + \langle z - u, v - z \rangle + \langle \rho N(w, y), \eta(v, z) \rangle \\
+ \rho [b(u, v) - b(u, z)] + \rho a(z, v - z)
\]
\[
\geq \psi(v, z), \quad \forall v, z \in H,
\]
and
\[
\psi(v, v) = 0, \quad \forall v \in H.
\]
So, condition (i) of Lemma 1.1 is satisfied.

From Remark 2.1 and Assumption 2.1(iv) we obtain that for each \( v \in H \), the mapping \( z \to \phi(v, z) \) is weakly upper semicontinuous on \( H \). And it is easy to show that for each fixed \( z \in H \), the set \( \{ v \in H : \psi(v, z) < 0 \} \) is convex. So, conditions (ii) and (iii) of Lemma 2.1 hold.

Set
\[
\omega = \| u \| + \rho \gamma \| u \| + \rho \delta \tau (\| w \| + \| y \|) + \rho \beta \| z \|,
\]
\[
K = \{ z \in H : \| z \| \leq \omega \}.
\]
Then \( K \) is a weakly compact subset of \( H \). It follows from Remark 2.1, Assumption 2.1 and the Lipschitz continuity of \( \eta \), that for given \( z \in H - K \), there exists \( v_0 = 0 \in K \) satisfying
\[
\psi(v_0, z) = \psi(0, z)
\]
\[
= \langle z - u, -z \rangle + \langle \rho N(w, y), \eta(0, z) \rangle + \rho [b(u, 0) - b(u, z)] \\
+ \rho a(z, -z)
\]
\[
= -\| z \|^2 + \langle u, z \rangle + \rho \langle N(w, y), \eta(0, z) \rangle \\
+ \rho b(u, -z) + \rho a(z, -z)
\]
\[
\leq -\| z \|^2 + \| u \| \| z \| + \rho \delta \tau (\| w \| + \| y \|) \| z \| \\
+ \rho \gamma \| u \| \| z \| + \rho \beta \| z \|^2
\]
\[
= -\| z \| [\| z \| - \| u \| - \rho \delta \tau (\| w \| + \| y \|) - \rho \gamma \| u \| - \rho \beta \| z \|)]
\]
\[
< 0.
\]
Hence, condition (iv) of Lemma 2.1 is satisfied. By virtue of Lemma 2.1, we conclude immediately that there exists some \( \bar{z} \in H \) such that
\[
\phi(v, \bar{z}) \geq 0, \quad \forall v \in H,
\]
that is, 
\[
\langle v - u, v - \bar{z} \rangle + \langle \rho N(w, y), \eta(v, \bar{z}) \rangle + \rho[b(u, v) - b(u, \bar{z})] \\
+ \rho a(\bar{z}, v - \bar{z}) \geq 0, \quad \forall v \in H.
\]
(3.2)

For arbitrary \( t \in (0, 1] \) and \( v \in H \), let \( x_t = tv + (1 - t)\bar{z} \). Replacing \( v \) by \( x_t \) in (3.2), we obtain
\[
0 \leq \langle x_t - u, x_t - \bar{z} \rangle + \langle \rho N(w, y), \eta(x_t, \bar{z}) \rangle + \rho[b(u, x_t) - b(u, \bar{z})] \\
+ \rho a(\bar{z}, x_t - \bar{z}) \\
= t\langle x_t - u, v - \bar{z} \rangle - \langle \rho N(w, y), \eta(\bar{z}, tv + (1 - t)\bar{z}) \rangle \\
+ \rho[b(u, tv + (1 - t)\bar{z}) - b(u, \bar{z})] + \rho a(\bar{z}, tv + (1 - t)\bar{z} - \bar{z}) \\
\leq t\langle x_t - u, v - \bar{z} \rangle - t\langle \rho N(w, y), \eta(\bar{z}, v) \rangle + \rho t[b(u, v) - b(u, \bar{z})] \\
+ \rho ta(\bar{z}, v - \bar{z}).
\]
Hence, we obtain
\[
\langle x_t - u, v - \bar{z} \rangle + \langle \rho N(w, y), \eta(v, \bar{z}) \rangle + \rho[b(u, v) - b(u, \bar{z})] \\
+ \rho a(\bar{z}, v - \bar{z}) \geq 0, \quad \forall v \in H.
\]

Letting \( t \to 0^+ \), we have
\[
\langle \bar{z} - u, v - \bar{z} \rangle + \langle \rho N(w, y), \eta(v, \bar{z}) \rangle + \rho[b(u, v) - b(u, \bar{z})] \\
+ \rho a(\bar{z}, v - \bar{z}) \geq 0, \quad \forall v \in H.
\]

Therefore \( \bar{z} \in H \) is a solution of the auxiliary problem (3.1). This completes the proof.

**Remark 3.1.** Theorem 3.1 generalizes Theorem 2.1 of Zeng et al. [11], Theorem 3.1 of Huang and Deng [5].

By using Theorem 3.1 and Nadler’s theorem[7], we suggest a new algorithm for the completely generalized set-valued strongly nonlinear mixed variational-like inequality problem (2.3).

**Algorithm 3.1.** For any \( u_0 \in H \), \( w_0 \in T(u_0) \) and \( y_0 \in A(u_0) \), compute \( \{w_n\}, \{y_n\}, \{u_n\} \) in \( H \), \( n = 0, 1, 2, \cdots \), by solving the auxiliary problem (3.1) with \( u = u_n \):
\[
w_n \in T(u_n), \quad \|w_n - w_{n+1}\| \leq (1 + \frac{1}{n + 1})D(T(u_n), T(u_{n+1})),
\]
(3.3)
\[
y_n \in A(u_n), \quad \|y_n - y_{n+1}\| \leq (1 + \frac{1}{n + 1})D(A(u_n), A(u_{n+1})),
\]
\[
\langle u_{n+1} - u_n, v - u_{n+1} \rangle + \langle \rho N(w_n, y_n), \eta(v, u_{n+1}) \rangle \\
+ \rho[b(u_n, v) - b(u_n, u_{n+1})] + \rho a(u_n, v - u_{n+1}) \geq 0, \quad \forall v \in H,
\]
(3.4)
where \( \rho > 0 \) is a constant.

4. Existence and convergence theorem
In this section, we prove the existence of a solution of the completely generalized set-valued strongly nonlinear mixed variational-like inequality problem (2.3) and the convergence of the sequences generalized by the algorithm 3.1.

**Theorem 4.1.** Let $N : H \times H \rightarrow H$ be Lipschitz continuous with respect to the first and second argument with Lipschitz constants $\zeta, \xi > 0$, respectively. Let $T, A : H \rightarrow CB(H)$ be Lipschitz continuous with constants $\nu, \mu > 0$, respectively, and $T$ be strongly monotone with respect to the first argument of $N$ with constant $\alpha > 0$. Let $\eta : H \times H \rightarrow H$ be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$. Let $a : H \times H \rightarrow \mathbb{R}$ be a continuous function which is linear in both arguments and satisfy (2.1)-(2.2). Assume that $b : H \times H \rightarrow \mathbb{R}$ satisfies properties (i)-(iv). Further, suppose that Assumption 2.1 holds and there exists a constant $\rho > 0$ such that

$$
(4.0) \quad \left| \rho - \frac{\delta - k}{\zeta^2 \nu^2 - k^2} \right| < \frac{\delta - k}{\zeta^2 \nu^2 - k^2},
$$

$$
\rho k < 1, \quad k < \delta, \quad k = \zeta \nu \sqrt{1 - 2 \sigma + \delta^2} + \xi \delta \mu + \gamma + \beta.
$$

Then there exist $u \in H$, $w \in T(u)$, and $y \in A(u)$ satisfying the completely generalized set-valued strongly nonlinear mixed variational-like inequality (2.3) and for $n \rightarrow \infty$,

$$
u_n \rightarrow u, \quad w_n \rightarrow w \quad y_n \rightarrow y,
$$

where the sequences $\{u_n\}, \{w_n\}, \{y_n\}$ are defined by Algorithm 3.1.

**Proof.** By (3.4), for any $v \in H$, we have

$$
\langle u_n - u_{n-1}, v - u_n \rangle + \langle \rho N(w_{n-1}, y_{n-1}), \eta(v, u_n) \rangle
$$

$$
+ \rho [b(u_{n-1}, v) - b(u_{n-1}, u_n)] + \rho a(u_{n-1}, v - u_n) \geq 0,
$$

$$
\langle u_{n+1} - u_n, v - u_{n+1} \rangle + \langle \rho N(w_n, y_n), \eta(v, u_{n+1}) \rangle
$$

$$
+ \rho [b(u_n, v) - b(u_n, u_{n+1})] + \rho a(u_n, v - u_{n+1}) \geq 0.
$$

Taking $v = u_{n+1}$ on (4.1) and $v = u_n$ in (4.2), respectively, we get

$$
\langle u_n - u_{n-1}, u_{n+1} - u_n \rangle + \langle \rho N(w_{n-1}, y_{n-1}), \eta(u_{n+1}, u_n) \rangle
$$

$$
+ \rho [b(u_{n-1}, u_{n+1}) - b(u_{n-1}, u_n)] + \rho a(u_{n-1}, u_{n+1} - u_n) \geq 0,
$$

$$
\langle u_{n+1} - u_n, u_n - u_{n+1} \rangle + \langle \rho N(w_n, y_n), \eta(u_n, u_{n+1}) \rangle
$$

$$
+ \rho [b(u_n, u_n) - b(u_n, u_{n+1})] + \rho a(u_n, u_n - u_{n+1}) \geq 0.
$$

Adding (4.3) and (4.4), we obtain

\[
\langle u_{n+1} - u_n, u_n - u_{n+1} \rangle \geq \langle u_n - u_{n-1}, u_n - u_{n+1} \rangle \\
- \rho \langle N(w_n, y_n) - N(w_{n-1}, y_{n-1}), \eta(u_n, u_{n+1}) \rangle \\
+ \rho b(u_{n-1} - u_n, u_n) + \rho b(u_n - u_{n-1}, u_{n+1}) \\
+ \rho a(u_n - u_{n-1}, u_{n+1} - u_n).
\]

Hence, we have

\[
\langle u_n - u_{n+1}, u_n - u_{n+1} \rangle \\
\leq \langle u_{n-1} - u_n, u_n - u_{n+1} \rangle - \rho \langle N(w_{n-1}, y_{n-1}) - N(w_n, y_n), \eta(u_n, u_{n+1}) \rangle \\
+ \rho |b(u_n - u_{n-1}, u_n) - b(u_n - u_{n-1}, u_{n+1})| \\
+ \rho a(u_{n-1} - u_n, u_{n+1} - u_n) \\
\leq \langle u_{n-1} - u_n, u_n - u_{n+1} \rangle - \rho \langle N(w_{n-1}, y_{n-1}) - N(w_n, y_n), \eta(u_n, u_{n+1}) \rangle \\
+ \rho \|N(w_{n-1}, y_{n-1}) - N(w_n, y_n), \eta(u_n, u_{n+1})\| \\
+ \rho b(u_n - u_{n-1}, u_n - u_{n+1}) + \rho a(u_{n-1} - u_n, u_{n+1} - u_n) \\
= \langle u_{n-1} - u_n - \rho \{N(w_{n-1}, y_{n-1}) - N(w_n, y_n)\}, u_n - u_{n+1} \rangle \\
+ \rho \langle N(w_{n-1}, y_{n-1}) - N(w_n, y_n), u_n - u_{n+1} - \eta(u_n, u_{n+1}) \rangle \\
+ \rho \langle N(w_n, y_n) - N(w_n, y_n), \eta(u_n, u_{n+1}) \rangle \\
+ \rho b(u_n - u_{n-1}, u_n - u_{n+1}) + \rho a(u_{n-1} - u_n, u_{n+1} - u_n),
\]

which implies

\[
\|u_n - u_{n+1}\|^2 \\
\leq \|u_{n-1} - u_n - \rho \{N(w_{n-1}, y_{n-1}) - N(w_n, y_n)\}\| \|u_n - u_{n+1}\| \\
+ \rho \|N(w_{n-1}, y_{n-1}) - N(w_n, y_n)\| \|u_n - u_{n+1} - \eta(u_n, u_{n+1})\| \\
+ \rho \|N(w_n, y_n) - N(w_n, y_n)\| \|\eta(u_n, u_{n+1})\| \\
+ \rho \gamma \|u_n - u_{n-1}\| \|u_n - u_{n+1}\| + \rho \beta \|u_{n-1} - u_n\| \|u_{n+1} - u_n\|.
\]

(4.5)

By the strong monotonicity of \(T\) with respect to the first argument of \(N\), the Lipschitz continuity of \(N\) with respect to the first and second argument, the \(D\)-Lipschitz continuities of the mappings \(T, A\), the strong monotonicity of
\( \eta \) and the Lipschitz continuity of \( \eta \), we obtain

\[
\|u_{n-1} - u_n - \rho \{N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1})\}\|^2 \\
= \|u_{n-1} - u_n\|^2 - 2\rho \{N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1}), u_{n-1} - u_n\} \\
+ \rho^2 \|N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1})\|^2 \\
\leq (1 - 2\rho \alpha)\|u_{n-1} - u_n\|^2 + \rho^2 \zeta^2 \|w_{n-1} - w_n\|^2 \\
\leq (1 - 2\rho \alpha)\|u_{n-1} - u_n\|^2 + \rho^2 \zeta^2 (1 + \frac{1}{n})^2 [D(T(u_{n-1}), T(u_n))]^2 \\
(4.6)
\leq [1 - 2\rho \alpha + \rho^2 \zeta^2 (1 + \frac{1}{n})^2 \nu^2]\|u_{n-1} - u_n\|^2,
\]

\[
\|N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1})\| \leq \zeta\|w_{n-1} - w_n\| \\
\leq \zeta (1 + \frac{1}{n})D(T(u_{n-1}), T(u_n)) \\
(4.7)
\leq \zeta (1 + \frac{1}{n})\nu\|u_{n-1} - u_n\|,
\]

\[
\|u_n - u_{n+1} - \eta(u_n, u_{n+1})\|^2 \\
= \|u_n - u_{n+1}\|^2 - 2\langle u_n - u_{n+1}, \eta(u_n, u_{n+1}) \rangle + \|\eta(u_n, u_{n+1})\|^2 \\
\leq (1 - 2\sigma + \delta^2)\|u_n - u_{n+1}\|^2, \\
(4.8)
\]

\[
\|N(w_n, y_n) - N(w_n, y_{n-1})\| \leq \xi\|y_n - y_{n-1}\| \\
\leq \xi (1 + \frac{1}{n})D(A(u_n), A(u_{n-1})) \\
(4.9)
\leq \xi (1 + \frac{1}{n})\mu\|u_n - u_{n-1}\|
\]

and

\[
\|\eta(u_n, u_{n+1})\| \leq \delta\|u_n - u_{n+1}\|. \\
(4.10)
\]

It follows from (4.5)-(4.10) that

\[
\|u_n - u_{n+1}\| \leq \sqrt{1 - 2\rho \alpha + \rho^2 \zeta^2 \nu^2 (1 + \frac{1}{n})^2 \|u_{n-1} - u_n\|} \\
+ \rho \zeta \nu (1 + \frac{1}{n}) \sqrt{1 - 2\sigma + \delta^2} \|u_{n-1} - u_n\| \\
+ \rho \xi \delta \mu (1 + \frac{1}{n})\|u_{n-1} - u_n\| + \rho \gamma \|u_{n-1} - u_n\| \\
+ \rho \beta\|u_{n-1} - u_n\|,
\]

that is,

\[
(4.11) \quad \|u_n - u_{n+1}\| \leq \theta_n\|u_{n-1} - u_n\|,
\]
where
\[
\theta_n = t_n(\rho) + \rho \zeta \nu \sqrt{1 - 2\sigma + \delta^2} + \rho \xi \delta \mu (1 + \frac{1}{n}) + \rho (\gamma + \beta)
\]
and
\[
t_n(\rho) = \sqrt{1 - 2\rho \alpha + \rho^2 \zeta^2 \nu^2 (1 + \frac{1}{n})^2}.
\]
Let
\[
\theta = t(\rho) + \rho \zeta \nu \sqrt{1 - 2\sigma + \delta^2} + \rho \xi \delta \mu + \rho (\gamma + \beta),
\]
and
\[
t(\rho) = \sqrt{1 - 2\rho \alpha + \rho^2 \zeta^2 \nu^2}.
\]
Clearly,
\[
t_n(\rho) \to t(\rho) \quad \text{and} \quad \theta_n \to \theta \quad \text{as} \quad n \to \infty.
\]
Note that
\[
\theta < 1 \Leftrightarrow t(\rho) + \rho k < 1,
\]
where \(k = \zeta \nu \sqrt{1 - 2\sigma + \delta^2} + \xi \delta \mu + \gamma + \beta\).

Now, it follows from the condition (4.0) that \(\theta < 1\). Hence, there is a positive number \(\theta_0 < 1\) and an integer \(n_0 \geq 1\) such that \(\theta_n \leq \theta_0 < 1\) for all \(n \geq n_0\).
Therefore, it follows from (4.11) that \(\{u_n\}\) is a Cauchy sequence in \(H\). Let \(u_n \to u\) as \(n \to \infty\). Since \(T\) and \(A\) are both \(D\)-Lipschitz continuous, by (3.3), we have
\[
\|w_n - w_{n+1}\| \leq (1 + \frac{1}{n + 1})D(T(u_n), T(u_{n+1})),
\]
\[
\leq (1 + \frac{1}{n + 1})\nu \|u_n - u_{n+1}\|,
\]
\[
\|y_n - y_{n+1}\| \leq (1 + \frac{1}{n + 1})D(A(u_n), A(u_{n+1})),
\]
\[
\leq (1 + \frac{1}{n + 1})\mu \|u_n - u_{n+1}\|.
\]
Therefore, \(\{w_n\}\) and \(\{y_n\}\) are also Cauchy sequences in \(H\). Let \(w_n \to w\) and \(y_n \to y\) as \(n \to \infty\). Since \(w_n \in T(u_n)\), we have
\[
d(w, T(u)) \leq \|w - w_n\| + d(w_n, T(u_n)) + D(T(u_n), T(u))
\]
\[
\leq \|w - w_n\| + \nu \|u_n - u\|
\]
\[
\to 0 \quad \text{as} \quad n \to \infty.
\]
Hence, we conclude that \(w \in T(u)\).
Similarly, we can obtain \( y \in A(u) \). Since \( N(w_n, y_n) \to N(w, y) \) strongly in \( H \), \( \eta(v, u_{n+1}) \to \eta(v, u) \) weakly in \( H \), \( b(u_n, u_{n+1}) \to b(u, u) \), \( b(u_n, v) \to b(u, v) \) and \( a(u_n, v - u_{n+1}) \to a(u, v - u) \), we deduce from (3.4) that
\[
\langle u - u, v - u \rangle + \langle \rho N(w, y), \eta(v, u) \rangle + \rho [b(u, v) - b(u, u)] + \rho a(u, v - u) \geq 0, \quad \forall v \in H,
\]
that is,
\[
\langle N(w, y), \eta(v, u) \rangle + b(u, v) - b(u, u) + a(u, v - u) \geq 0, \quad \forall v \in H.
\]
This completes the proof.

**Remark 4.1.** If \( a(u, v - u) = 0 \) for all \( u, v \in H \), then Theorem 4.1 reduces to Theorem 3.1 of [11] and Theorem 4.1 of [5].

**REFERENCES**


**Received:** March 10, 2008