

Two Loop Soliton Solutions to the Reduced Ostrovsky Equation¹

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Abstract

The bifurcation method of planar systems and simulation method of differential equations are employed to investigate loop soliton solutions of the reduced Ostrovsky equation (ROE). The parameter representation of loop soliton solutions of the ROE are obtained. The planar graphs of the loop soliton solutions is shown under the some parameter. These results are supplement to investigate the ROE.

Keywords: Reduced Ostrovsky equation ; Phase portrait; Loop soliton; Numerical simulation

1 Introduction

The nonlinear dispersive equation

$$(u_t + c_0 u_x + \alpha u u_x + \beta u_{xxx})_x = \gamma u \quad (1.1)$$

is a model for weakly nonlinear surface and internal waves in a rotating ocean, where c_0 is the velocity of dispersionless linear waves, α is the nonlinear coefficient, β and γ are the dispersion coefficients. It was derived by In Ostrovsky in 1978 [1] and is now known as Ostrovsky equation.

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For long waves, for which high-frequency dispersion is negligible, $\beta = 0$ and (1.1) becomes the so-called reduced Ostrovsky equation (ROE), namely

$$(u_t + c_0 u_x + \alpha u u_x)_x = \gamma u. \quad (1.2)$$

This equation was considered for the first time in the original paper by Ostrovsky [1]. A thorough discussion of its travelling-wave solutions only appeared recently in a paper [2] by Stepanyants. The present paper complements Stepanyants' work by giving some periodic solutions of the (1.2) in explicit analytical form rather than by just referring to the first integral of (1.2) as was done in [1,2] for example.

In [1,3] it was shown that the (1.2) has an analytical limiting periodic-wave solution that has a discontinuous slope at each crest and is parabolic between crests. Boyd [4,5] dubbed these maximum amplitude waves corner waves or parabolic waves. In [6] it is shown that the VE and the OHE are particular forms of the ROE, explicit analytical periodic and solitary travelling-wave solutions of the OHE are obtained.

Here, Our aim in this paper is to use the bifurcation method of planar systems and simulation method of differential equations [7,8] to investigate the loop soliton wave solutions of the (1.2). The parameter representation of loop soliton solutions of the ROE are obtained. The planar graphs of the loop soliton solutions is shown under the some parameter. These results include the loop soliton solutions of the ROE in [6].

The rest of this paper is organized as follows. In Section 2, we study the bifurcations of phase portrait. In Section 3, We derive the parameter representations of loop soliton solutions and make the numerical simulation for loop soliton. Finally, a short conclusion is given in section 4.

2 Bifurcation Phase Portrait

In this section we derive (1.2), the traveling wave system and study its bifurcation phase portrait. Observe that the case $\alpha = 0$ is much simpler than the one we are considering, so we shall assume $\alpha \neq 0$.

Substituting $u(x, t) = \phi(\xi)$ with $\xi = x - ct$ in (1.2), we have

$$\alpha(\phi')^2 + (c_0 - c + \alpha\phi)\phi'' = \gamma\phi, \quad (2.1)$$

where c is the wave speed, which is called the traveling wave equation.

Letting $\phi' = y$, we obtain a planar system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\gamma\phi - \alpha y^2}{\alpha\phi + c_0 - c}, \quad (2.2)$$

which is called traveling wave system. Our aim is to study the phase portrait of system (2.2). But system (2.2) has a singular line $\phi = q = \frac{v}{\alpha}$, which is inconvenient to our study, where $v = c - c_0$. So we make the transformation

$$d\xi = \alpha(\phi - q)d\tau. \quad (2.3)$$

Thus system (2.2) becomes a Hamiltonian system

$$\frac{d\phi}{d\tau} = \alpha(\phi - q)y, \quad \frac{dy}{d\tau} = \gamma\phi - \alpha y^2. \quad (2.4)$$

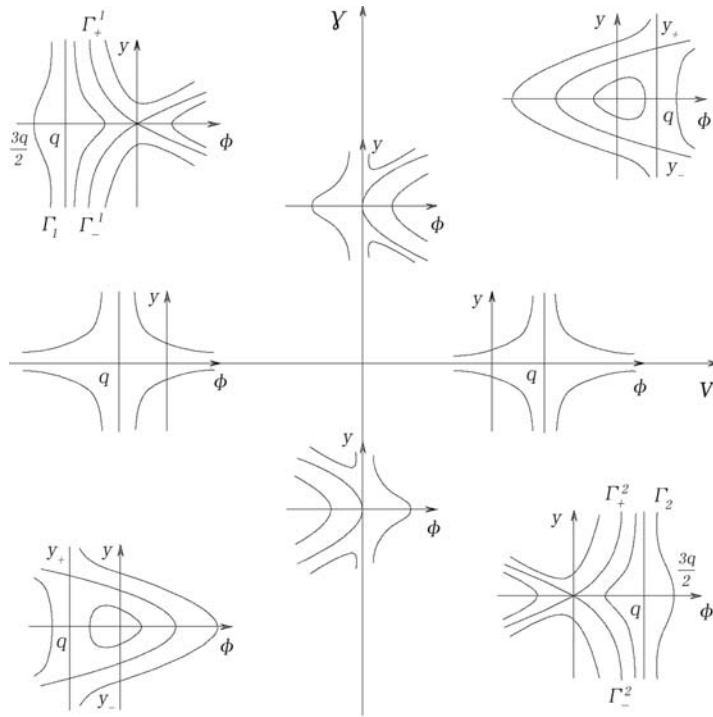
Thus systems (2.2) and (2.4) have the same first integral

$$H(\phi, y) = (\phi - q)^2(y^2 - \frac{\gamma}{3\alpha}(2\phi + q)) = h. \quad (2.5)$$

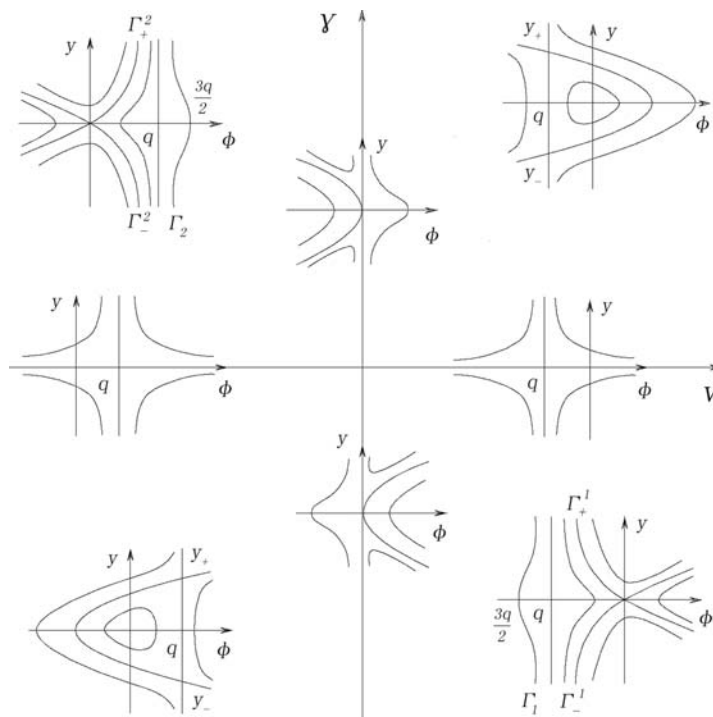
Therefore both systems (2.2) and (2.4) have the same topological phase portraits except the straight line $\phi = q$.

Obviously, from the bifurcation method of planar systems, systems (2.4) have three singular points $(0,0)$ and (q, y_{\pm}) when $\gamma v > 0$, where $y_{\pm} = \pm\sqrt{\frac{\gamma v}{\alpha^2}} = \pm\sqrt{\frac{\gamma q}{\alpha}}$, and $(0,0)$ is a center point, (q, y_{\pm}) are two saddle point. Systems (2.4) have only one singular points $(0,0)$ when $\gamma v \leq 0$, and it is a saddle point when $\gamma \neq 0$ or $\gamma = 0$ and $v = 0$, it is a degenerate saddle point when $\gamma \neq 0$ and $v = 0$.

According to the above analysis, we draw the bifurcation phase portrait of (2.2) and (2.4), shown in Fig.1.



a. $\alpha > 0$.



b. $\alpha < 0$.

Fig.1 The bifurcation phase portrait of systems (2.2) and (2.4).

3 Loop Soliton Solutions

From the derivation in Sec.2 we notice that the curves defined by $H(\phi, y) = H(0, 0) = -\frac{\gamma q^3}{3\alpha}$ correspond to 6 different orbits of (2.4) consisting of two stable manifolds, two unstable manifolds of the saddle point $(0, 0)$ and two open curves passing through the point $(\frac{3q}{2}, 0)$ (see Fig.1 $\Gamma_{\pm}^1, \Gamma_1, \Gamma_{\pm}^2$ and Γ_2). From [7] ideal the Γ_{\pm}^1 and Γ_1 correspond to a loop soliton, Γ_{\pm}^2 and Γ_2 correspond to other loop soliton.

Case 1. Under the conditions $\alpha > 0, \gamma > 0$ and $v < 0$ or $\alpha < 0, \gamma < 0$ and $v > 0$, taking $h = -\frac{\gamma q^3}{3\alpha}$, (2.5) can be reduced to

$$y = \pm \sqrt{\frac{\gamma \phi^2 (2\phi - 3q)}{3\alpha (\phi - q)^2}}. \quad (3.1)$$

Taking $(\frac{3q}{2}, 0)$ as initial condition, then substituting (3.1) into the left expression of (2.4) and integrating it, we have

$$\int_{\frac{3q}{2}}^{\phi} \frac{ds}{s\sqrt{2s-3q}} = \pm \sqrt{\frac{\alpha\gamma}{3}} \int_0^{\tau} ds. \quad (3.2)$$

From (3.2), we get

$$\phi = \frac{3q}{2} \operatorname{sech}^2(a\tau), \quad (3.3)$$

where $a = \frac{\sqrt{-\alpha\gamma q}}{2}$. Substituting (3.3) into (2.3) and integrating it, we have

$$\int_0^{\xi} ds = \int_0^{\tau} \alpha \left(\frac{3q}{2} \operatorname{sech}^2(as) - q \right) ds. \quad (3.4)$$

From (3.4), we get

$$\xi = \alpha q \left(\frac{3}{2a} \tanh(a\tau) - \tau \right). \quad (3.5)$$

Thus we obtain a loop soliton solution

$$\phi = \frac{3q}{2} \operatorname{sech}^2 a\tau, \quad \xi = \alpha q \left(\frac{3}{2a} \tanh(a\tau) - \tau \right), \quad (3.6)$$

where $\frac{3q}{2} \leq \phi < 0, -\infty < \xi < +\infty$.

Case 2. Under the conditions $\alpha > 0, \gamma < 0$ and $v > 0$ or $\alpha < 0, \gamma > 0$ and $v < 0$, taking $h = -\frac{\gamma q^3}{3\alpha}$, (2.5) can be reduced to

$$y = \pm \sqrt{-\frac{\gamma \phi^2 (3q - 2\phi)}{3\alpha (\phi - q)^2}}. \quad (3.7)$$

Taking $(\frac{3q}{2}, 0)$ as initial condition, then substituting (3.7) into the left expression of (2.4) and integrating it, we have

$$\int_{\phi}^{\frac{3q}{2}} \frac{ds}{s\sqrt{3q-2s}} = \pm \sqrt{-\frac{\alpha\gamma}{3}} \int_{\tau}^0 ds. \quad (3.8)$$

From (3.8), we get

$$\phi = \frac{3q}{2} \operatorname{sech}^2(a\tau). \quad (3.9)$$

Substituting (3.9) into (2.3) and integrating it, we have

$$\int_0^{\xi} ds = \int_0^{\tau} \alpha \left(\frac{3q}{2} \operatorname{sech}^2(as) - q \right) ds. \quad (3.10)$$

From (3.10), we get

$$\xi = \alpha q \left(\frac{3}{2a} \tanh(a\tau) - \tau \right). \quad (3.11)$$

Thus we obtain other loop soliton solution

$$\phi = \frac{3q}{2} \operatorname{sech}^2 a\tau, \quad \xi = \alpha q \left(\frac{3}{2a} \tanh(a\tau) - \tau \right), \quad (3.12)$$

where $0 < \phi \leq \frac{3q}{2}$, $-\infty < \xi < +\infty$.

From above analysis, we have the following conclusion.

Proposition 1. (1) For given $\alpha > 0, \gamma > 0$ and $v < 0$ or $\alpha < 0, \gamma < 0$ and $v > 0$, then (1.2) has a loop soliton, (See Fig.2(a)) and has parameter representation (3.6).

(2) For given $\alpha > 0, \gamma < 0$ and $v > 0$ or $\alpha < 0, \gamma > 0$ and $v < 0$, then (1.2) has other loop soliton, (See Fig.2(b)) and has parameter representation (3.12).

Example 1. When $c_0 = 1, c = 0.5, \alpha = 1$ and $\gamma = 1$ then $v = -0.5, q = -0.5$ and $\frac{3q}{2} = -0.75$. Choosing $\phi(0) = -0.75$, from (3.6) we can simulate the loop soliton (1.2) as (a) in Fig.2.

Example 2. When $c_0 = 0.5, c = 1, \alpha = 1$ and $\gamma = -1$ then $v = 0.5, q = 0.5$ and $\frac{3q}{2} = 0.75$. Choosing $\phi(0) = 0.75$, from (3.12) we can simulate the loop soliton of (1.2) as (b) in Fig.2.

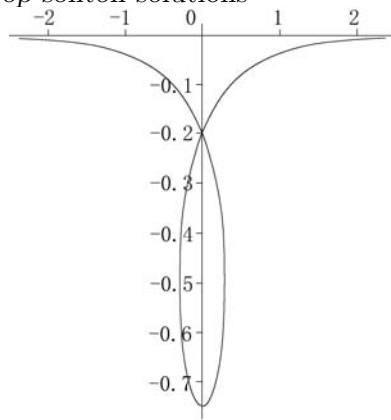
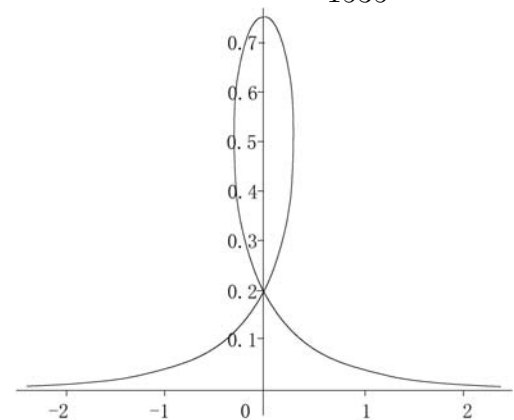
(a) $\phi(0) = -0.75$.(b) $\phi(0) = 0.75$.

Fig.2. The simulate of the loop soliton of (1.2), when (a) $c_0 = 1, c = 0.5, \alpha = 1$ and $\gamma = 1$, (b) $c_0 = 0.5, c = 1, \alpha = 1$ and $\gamma = -1$.

4 Conclusion

In this paper, we have studied the bifurcation and global behavior a reduced Ostrovsky equation, and obtained the condition under which the loop soliton solutions appear. The planar graphs of the loop soliton are simulated in Fig.2. Finally the parameter representations of loop soliton solutions be obtained. Numerical simulation results show the consistence with the theoretical analysis.

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