An Error Bound on Uniform Approximation of Bounded Function by Bernstein Polynomial

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Abstract

Let \( f : [0, 1]^p \rightarrow \mathbb{R}^q \) be a bounded function. In this paper, we used technique from [11] to give a bound on uniform approximation of bounded function \( f \) by Bernstein polynomial. The bound is of the form \( C \omega \left( \frac{1}{\sqrt{n}} \right) \) where \( C \) is a constant and \( \omega \left( \frac{1}{\sqrt{n}} \right) \) is a modulus of \( f \) depend on \( \frac{1}{\sqrt{n}} \).

Keywords: Bernstein polynomial, uniform approximation, Weierstrass approximation, modulus of function

1 Introduction

In this paper, we investigated a uniform bound on the approximation of bounded function \( f = (f_1, f_2, ..., f_p) : [0, 1]^p \rightarrow \mathbb{R}^q \) by Bernstein polynomial \( \tilde{B}_n(f, \cdot) : [0, 1]^p \rightarrow \mathbb{R}^q \) which is defined by

\[
\tilde{B}_n(f, \cdot) = (B_n(f_1, \cdot), B_n(f_2, \cdot), \ldots, B_n(f_q, \cdot))
\]

where \( B_n(f_k, \cdot) : [0, 1]^p \rightarrow \mathbb{R} \) define by

\[
B_n(f_k, t) = \sum_{j_1+j_2+...+j_{p+1}=n} \frac{p_{j_1}}{n} \frac{p_{j_2}}{n} \ldots \frac{p_{j_{p+1}}}{n} \left( \frac{n}{j_1, j_2, \ldots, j_{p+1}} \right) T_1^{j_1} T_2^{j_2} \ldots T_{p+1}^{j_{p+1}}
\]

(1.1)
for $\vec{t} = (t_1, t_2, ..., t_p)$, $T_i = \frac{t_i}{p}, i = 1, 2, ..., p, T_{p+1} = 1 - (T_1 + T_2 + ... + T_p)$ and
\[
\binom{n}{j_1, j_2, ..., j_{p+1}} = \frac{n!}{j_1! j_2! ... j_{p+1}!}.
\]
Observe that, in case of $p = q = 1$,
\[
\tilde{B}_n(f, t) = \sum_{j=0}^{n} \binom{n}{j} t^j (1-t)^{n-j} f\left(\frac{j}{n}\right)
\]
(1.2)
is the Bernstein polynomial function which defined by Bernstein([8]). We note that Bernstein polynomial are useful in Bayesian statistics because of their interpretation as mixtures of Beta distribution ([6], [9]). The original work is back to 1912, when Bernstein ([8]) gave the well-known Weierstrass approximation theorem.

**Theorem 1.1. (Weierstrass Approximation Theorem)**

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and $\tilde{B}_n(f, \cdot) : [0, 1] \to \mathbb{R}$ be Bernstein polynomial which defined by (1.2). Then for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that
\[
|f(t) - \tilde{B}_n(f, t)| < \epsilon
\]
for every $t \in [0, 1]$.

Since then, there are many authors investigate a bound of this approximation such as Popoviciu ([7]) , 1934 , Kac ([4],[5]) in 1938 - 1939 , Dallal and Hall ([9]) in 1983 , Diaconics and Ylvisaker ([6]) in 1985.

Until 1997 Both Gzyl and Palacios ([1]) investigated a bound of the approximation in case of $f$ is Lipschitz function i.e., there exists a constant $L$ such that $|f(x) - f(y)| \leq L |x - y|$ for any $x, y \in [0, 1]$. They yield the rate $\sqrt{\frac{\ln n}{n}}$ as follows:
\[
|f(t) - \tilde{B}_n(f, t)| \leq K \sqrt{\frac{\ln n}{n}} \text{ for all } t \in [0, 1].
\]

In case of multi-dimension, K.Neammanee and S.Sirisub([3]) used the probabilistic tools to approximate and they showed that for $\epsilon > 0$,
\[
\|f(\vec{t}) - \tilde{B}_n(f, \vec{t})\| \leq \epsilon \text{ for all } \vec{t} \in [0, 1]^p.
\]
(1.3)

In (1.3) K.Neammanee and S.Sirisub, they did not give bound of the approximation. In 2001, K.Neammanee([2]) gave a bound of this approximation.
in case of Lipschitz function. He showed that

\[ \| f(\hat{t}) - \tilde{B}_n(f, \hat{t}) \| \leq K \sqrt{\frac{\ln n}{n}} \text{ for } \hat{t} \in [0, 1]^p. \] (1.4)

A bound in (1.4) was improved in 2004 by Y.Pankla and E.Suntonsinsongvon([11]). Let \( f : \mathbb{R}^p \to \mathbb{R}^q \) be holder function with exponent \( \alpha > 0 \), i.e., there exists a constant \( C \) such that \( \| f(x) - f(y) \| \leq C \| x - y \|^\alpha \) for every \( x, y \in [0, 1]^p \). Then there exists a constant \( K \) depend on \( f \) such that

\[ \| f(\hat{t}) - \tilde{B}_n(f, \hat{t}) \| \leq \frac{K}{n^{\alpha/2}} \text{ for } \hat{t} \in [0, 1]^p \text{ and } n \in \mathbb{N}. \] (1.5)

In this paper we will relieve the condition of Y.Punkla and E.Sontonsinsongvon to a bounded function. Theorem 3 is our main result.

**Theorem 1.2.** Let \( f : [0, 1]^p \to \mathbb{R}^q \) be a bounded function, then there exists a constant \( C > 0 \) such that

\[ \| f(\hat{t}) - \tilde{B}_n(f, \hat{t}) \| \leq C \omega\left(\frac{1}{\sqrt{n}}\right) \text{ for all } \hat{t} \in [0, 1]^p \]

where the modulus \( \omega(\delta) \) of \( f \) is defined for \( \delta > 0 \) by

\[ \omega(\delta) = \sup_{\| \hat{t}_1 - \hat{t}_2 \| \leq \delta} \left\| f(\hat{t}_1) - f(\hat{t}_2) \right\|. \]

Note that the modulus of \( f \) depends on \( \delta \), \( f \) and the interval \([0, 1]^p\). So that \( \omega(\delta) \) is shorthand for \( \omega(f; [0, 1]^p; \delta) \). If \( f \) is holder with exponent \( \alpha > 0 \), we have \( \omega\left(\frac{1}{\sqrt{n}}\right) = \frac{C}{n^{\alpha/2}} \). Hence Theorem 1.2 generizes (1.5).
2 Proof of Main Results

By the fact that
\[
\left\| f(\tilde{t}) - \bar{B}_n(f, \tilde{t}) \right\|
= \left\| (f_1(\tilde{t}), f_2(\tilde{t}), \ldots, f_p(\tilde{t})) - (B_n(f_1, \tilde{t}), B_n(f_2, \tilde{t}), \ldots, B_n(f_p, \tilde{t})) \right\|
= \sqrt{(f_1(\tilde{t}) - B_n(f_1, \tilde{t}))^2 + (f_2(\tilde{t}) - B_n(f_2, \tilde{t}))^2 + \ldots + (f_p(\tilde{t}) - B_n(f_p, \tilde{t}))^2}
\leq \sqrt{\left( f_1(\tilde{t}) - B_n(f_1, \tilde{t}) \right)^2 + \left( f_2(\tilde{t}) - B_n(f_2, \tilde{t}) \right)^2 + \ldots + \left( f_p(\tilde{t}) - B_n(f_p, \tilde{t}) \right)^2}
= \sum_{k=1}^{q} \left| f_k(\tilde{t}) - B_n(f_k, \tilde{t}) \right|, \quad (2.1)
\]
it suffices to prove that for each \( k = 1, 2, \ldots, p \), there exists a positive constant \( C \) such that
\[
\left| f_k(\tilde{t}) - B_n(f_k, \tilde{t}) \right| \leq C \omega\left( \frac{1}{\sqrt{n}} \right).
\]
By the fact that
\[
1 = \sum_{j_1+j_2+\ldots+j_{p+1}=n} \left( \begin{array}{c} n \\ j_1, j_2, \ldots, j_{p+1} \end{array} \right) T_1^{j_1} T_2^{j_2} \ldots T_{p+1}^{j_{p+1}}
\]
for \( \tilde{t} = (t_1, t_2, \ldots, t_p) \in [0, 1]^p, T_i = \frac{t_i}{p}, i = 1, 2, \ldots, p \) and \( T_{p+1} = 1 - (T_1 + T_2 + \ldots + T_p) \) (eq.(1) of [11]), we have
\[
f_k(\tilde{t}) = \sum_{j_1+j_2+\ldots+j_{p+1}=n} f_k(\tilde{t}) \left( \begin{array}{c} n \\ j_1, j_2, \ldots, j_{p+1} \end{array} \right) T_1^{j_1} T_2^{j_2} \ldots T_{p+1}^{j_{p+1}}.
\]
Hence
\[
\left| f_k(\tilde{t}) - B_n(f_k, \tilde{t}) \right|
= \left| \sum_{j_1+j_2+\ldots+j_{p+1}=n} \left( f_k(\tilde{t}) - f_k\left( \frac{pj_1}{n}, \frac{pj_2}{n}, \ldots, \frac{pj_p}{n} \right) \right) \left( \begin{array}{c} n \\ j_1, j_2, \ldots, j_{p+1} \end{array} \right) T_1^{j_1} T_2^{j_2} \ldots T_{p+1}^{j_{p+1}} \right|
\leq \sum_{j_1+j_2+\ldots+j_{p+1}=n} \left| f_k(\tilde{t}) - f_k\left( \frac{pj_1}{n}, \frac{pj_2}{n}, \ldots, \frac{pj_p}{n} \right) \right| \left( \begin{array}{c} n \\ j_1, j_2, \ldots, j_{p+1} \end{array} \right) T_1^{j_1} T_2^{j_2} \ldots T_{p+1}^{j_{p+1}}
= \sum_{j_1+j_2+\ldots+j_{p+1}=n} \omega(\frac{\sqrt{\left| \tilde{t} - \left( \frac{pj_1}{n}, \frac{pj_2}{n}, \ldots, \frac{pj_p}{n} \right) \right|}}{n}) \left( \begin{array}{c} n \\ j_1, j_2, \ldots, j_{p+1} \end{array} \right) T_1^{j_1} T_2^{j_2} \ldots T_{p+1}^{j_{p+1}}.
\quad (2.2)
Note that 

\[ \sum_{j_1+j_2+\ldots+j_{p+1}=n} j_i \left( \begin{array}{c} n \\ j_1, j_2, \ldots, j_{p+1} \end{array} \right) T_1^{j_1} T_2^{j_2} \ldots T_p^{j_p+1} = n T_1 \]

and 

\[ \sum_{j_1+j_2+\ldots+j_{p+1}=n} j_i \left( \begin{array}{c} n \\ j_1, j_2, \ldots, j_{p+1} \end{array} \right) T_1^{j_1} T_2^{j_2} \ldots T_p^{j_p+1} = n(n-1) T_1^2 + n T_1 \]

for \( i = 1, 2, \ldots, p + 1 \) (eq. (4) and (5) of [11]).

Hence 

\[
\sum_{j_1+j_2+\ldots+j_{p+1}=n} \left\| \hat{t} - \left( \frac{pj_1}{n}, \frac{pj_2}{n}, \ldots, \frac{pj_p}{n} \right) \right\|^2 \left( \begin{array}{c} n \\ j_1, j_2, \ldots, j_{p+1} \end{array} \right) T_1^{j_1} T_2^{j_2} \ldots T_p^{j_p+1} \right\|^2 
\]

\[
= \left[ \sum_{j_1+j_2+\ldots+j_{p+1}=n} \left( \frac{nj_1}{n}, \frac{nj_2}{n}, \ldots, \frac{nj_p}{n} \right) \right] T_1^{j_1} T_2^{j_2} \ldots T_p^{j_p+1} \right\|^2 
\]

\[
= \sum_{i=1}^{p} \sum_{j_1+j_2+\ldots+j_{p+1}=n} \left( T_i - \frac{j_i}{n} \right)^2 \left( \begin{array}{c} n \\ j_1, j_2, \ldots, j_{p+1} \end{array} \right) T_1^{j_1} T_2^{j_2} \ldots T_p^{j_p+1} \right\|^2 
\]

\[
= p \left[ \sum_{i=1}^{p} \sum_{j_1+j_2+\ldots+j_{p+1}=n} \left( T_i^2 - \frac{j_i^2}{n} \right) \left( \begin{array}{c} n \\ j_1, j_2, \ldots, j_{p+1} \end{array} \right) T_1^{j_1} T_2^{j_2} \ldots T_p^{j_p+1} \right\|^2 
\]

\[
= \frac{p}{n^{\frac{p}{2}}} \left[ \sum_{i=1}^{p} T_i (1 - T_i) \right] \right\|^2 
\]

\[
\leq C \frac{1}{n^{1/2}}. 
\]

By the fact that \( \omega(\lambda \delta) \leq (1 + \lambda) \omega(\delta) \) for \( \lambda > 0 \) ([10], pp. 15), we have 

\[
\omega\left( \left\| \hat{t} - \left( \frac{pj_1}{n}, \frac{pj_2}{n}, \ldots, \frac{pj_p}{n} \right) \right\| \right) = \omega(n^\frac{1}{2} \left\| \hat{t} - \left( \frac{pj_1}{n}, \frac{pj_2}{n}, \ldots, \frac{pj_p}{n} \right) \right\| \left\| n^{\frac{1}{2}} \right\|)
\]

\[
= (1 + n^{\frac{1}{2}}) \left\| \hat{t} - \left( \frac{pj_1}{n}, \frac{pj_2}{n}, \ldots, \frac{pj_p}{n} \right) \right\| \omega\left( \frac{1}{\sqrt{n}} \right). 
\]

\[
(2.4) 
\]
Therefore, by (2.2)-(2.4),

\[
|f_k(\vec{t}) - B_{nk}(\vec{t})| \\ \leq \omega\left(\frac{1}{\sqrt{n}}\right) \sum_{j_1+j_2+\ldots+j_{p+1}=n} \left(\begin{array}{c} n \\ j_1, j_2, \ldots, j_{p+1} \end{array}\right) T_1^{j_1} T_2^{j_2} \ldots T_{p+1}^{j_{p+1}} \\
+ n^{1/2} \sum_{j_1+j_2+\ldots+j_{p+1}=n} \left\| \vec{t} - \left(p_{j_1} \frac{n}{n}, p_{j_2} \frac{n}{n}, \ldots, p_{j_{p+1}} \frac{n}{n}\right) \right\| \left(\begin{array}{c} n \\ j_1, j_2, \ldots, j_{p+1} \end{array}\right) T_1^{j_1} T_2^{j_2} \ldots T_{p+1}^{j_{p+1}}
\]

\leq C\omega\left(\frac{1}{\sqrt{n}}\right). \text{ Then the theorem is proved.}

References


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