

Modified Model for Finding Unique Optimal Solution in Data Envelopment Analysis

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Abstract

In Data Envelopment Analysis (DEA) models which were developed so far, the difficulties arise from alternative optimal solutions in calculating the value of θ . To remove this difficulty a perturbation method is used to find a unique solution and the results seems to be promising.

Keywords: Linear Programming, Data Envelopment Analysis, Efficiency

1 Introduction

Data Envelopment Analysis (DEA) has opened up several interesting possibilities for the estimation of production correspondences. This approach does not require any stringent assumptions about the underlying production correspondences, unlike the classical approaches for estimating production functions which necessarily assume several restrictive properties for the production correspondence that are implicit in the use of a prespecified parametric form for the estimation.

Data Envelopment Analysis, introduced by Charnes, Cooper and Rhodes (CCR) (1978) and further formalized by Banker, Charnes and Cooper (BCC)

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(1984), provides a nonparametric approach for estimating ‘production’ technologies and hence measuring inefficiencies in production. The BCC ratio presents technical inefficiency and the CCR ratio comprehends both technical and scale inefficiencies via the optimal value of the ratio form as obtained directly from the data without requiring *a priori* specification of weights and/or explicit delineation of assumed functional forms of relation between inputs and outputs.

The second section in the paper represents a brief discussion about Data envelopment Analysis models. In section 3, the modified model briefly is discussed and the last section presents the conclusion.

2 Models of DEA

Consider a Decision Making Unit (DMU) that consumes $X = (X_1, X_2, \dots, X_m)$ inputs to produce $Y = (Y_1, Y_2, \dots, Y_s)$ outputs. The efficiency of DMU_{*p*} is obtained by solving the following linear programming problem:

$$\begin{aligned}
 & \text{Min} && \theta \\
 & \text{S.t} && \sum_{j=1}^n \lambda_j x_{ij} - \theta x_{ij} \leq 0 \quad , \quad i = 1, 2, \dots, m, \\
 & && \sum_{j=1}^n \lambda_j y_{rj} \geq y_{rp} \quad , \quad r = 1, 2, \dots, s, \\
 & && \lambda_j \geq 0 \quad , \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{1}$$

It’s dual is as follow:

$$\begin{aligned}
 & \text{Max} && e_p = \sum_{r=1}^s u_r y_{rp} \\
 & \text{S.t} && \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad , \quad j = 1, \dots, n, \\
 & && \sum_{i=1}^m v_i x_{ip} = 1, \\
 & && u_r, v_i \geq 0 \quad , \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{2}$$

This model is called CCR input orientation multiplier side problem.

In 1979 CCR [6] introduced the following model; which uses ϵ (where ϵ is a positive non-Archimedean infinitesimal) to prevent of vanishing some U and V .

$$\begin{aligned}
 \text{Max} \quad & e_p = \sum_{r=1}^s u_r y_{rp} \\
 \text{S.t} \quad & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad , \quad j = 1, \dots, n, \\
 & \sum_{i=1}^m v_i x_{ip} = 1, \\
 & u_r, v_i \geq \epsilon \quad , \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{3}$$

The dual of the problem (3), in this case, will be as follows:

$$\begin{aligned}
 \text{Min} \quad & \theta - \epsilon \left(\sum_{r=1}^s s_r^+ + \sum_{i=1}^m s_i^- \right) \\
 \text{S.t} \quad & \sum_{j=1}^n \lambda_j x_{ij} - \theta x_{ij} + s_i^- = 0 \quad , \quad i = 1, 2, \dots, m, \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{rp} \quad , \quad r = 1, 2, \dots, s, \\
 & \lambda_j \geq 0 \quad , \quad j = 1, 2, \dots, n, \\
 & s_r^+ \geq 0 \quad , \quad r = 1, 2, \dots, s, \\
 & s_i^- \geq 0 \quad , \quad i = 1, 2, \dots, m.
 \end{aligned} \tag{4}$$

Definition: DMU_p is efficient if and only if:

- 1) $\theta^* = 1$
- 2) $s_r^+ = 0$ and $s_i^- = 0$, $r = 1, 2, \dots, s$, $i = 1, 2, \dots, m$.

3 Modified model

Four DMUs are consider with two inputs and two outputs as follows:

	DMU ₁	DMU ₂	DMU ₃	DMU ₄
INPUT ₁	1	1	1	1
INPUT ₂	2	3	3	1
OUTPUT ₁	4	6	6	2
OUTPUT ₂	1	2	3	2

By using (1) for DMU₁, the following alternative optimal solutions are obtained. In the first optimal solution, DMU₁ is efficient, and in the other one, it is not.

θ^*	λ_1^*	λ_2^*	λ_3^*	λ_4^*	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}
1	1	0	0	0	0	0	0	0
1	0	0	0.5	0.5	0	0	0	1.5

Theorem(1):

Consider the following linear programming:

$$\begin{aligned} &Min \quad CX \\ &S.t \quad AX = b \\ &\quad \quad X \geq 0. \end{aligned}$$

there is ϵ^* , so that for each $0 < \epsilon < \epsilon^*$, the following linear programming is total non-degenerate.

$$\begin{aligned} &Min \quad CX \\ &S.t \quad AX = b(\epsilon) \\ &\quad \quad X \geq 0. \end{aligned}$$

where $(b(\epsilon))_i = b_i + \epsilon^i$ (for proof see [7]).

Theorem(2):

Consider a linear programming and its dual as follows:

$$\begin{array}{ll} P : & Min \quad CX \\ & S.t \quad AX \geq b \\ & \quad \quad X \geq 0. \end{array} \qquad \begin{array}{ll} D : & Max \quad Yb \\ & S.t \quad YA \leq b \\ & \quad \quad Y \geq 0. \end{array}$$

If problem (P) has alternative optimal solution, then problem (D) has degenerate optimal solution (for proof see [3]).

By theorems (1) and (2), we have, if primal or dual problem are total non-degenerate, then others poses unique optimal solution. so (4) is perturbed so that the problem is total non-degenerate. The modified model is as follows:

$$\begin{aligned}
 \text{Max} \quad & e_p = \sum_{r=1}^s u_r y_{rp} \\
 \text{S.t} \quad & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq \epsilon^j, \quad j = 1, \dots, n, \\
 & \sum_{i=1}^m v_i x_{ip} = 1 + \epsilon^{n+1}, \\
 & u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{5}$$

and its dual is as follows:

$$\begin{aligned}
 \text{Min} \quad & z_p = \theta(1 + \epsilon^{n+1}) + \sum_{j=1}^n \epsilon^j \lambda_j - \epsilon \left(\sum_{r=1}^s s_r^+ + \sum_{i=1}^m s_i^- \right) \\
 \text{S.t} \quad & \sum_{j=1}^n \lambda_j x_{ij} - \theta x_{ij} + s_i^- = 0, \quad i = 1, 2, \dots, m, \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{rp}, \quad r = 1, 2, \dots, s, \\
 & \lambda_j \geq 0, \quad j = 1, 2, \dots, n, \\
 & s_r^+ \geq 0, \quad r = 1, 2, \dots, s, \\
 & s_i^- \geq 0, \quad i = 1, 2, \dots, m.
 \end{aligned} \tag{6}$$

By theorems (1) and (2) there is ϵ^* , so that for each $0 < \epsilon < \epsilon^*$ this problem has not multiple optimal solution.

DMU_p is efficient if and only if:

- 1) $z_p^* = 1$
- 2) $s_r^+ = 0$ and $s_i^- = 0$, $r = 1, 2, \dots, s$, $i = 1, 2, \dots, m$.

An example is solved by using this model for DMU₁ and the result has been presented in tableu (1), which shows that DMU₁ is not efficient.

θ^*	λ_1^*	λ_2^*	λ_3^*	λ_4^*	s_1^-	s_2^-	s_1^+	s_2^+
1	0	0	0	0.6666	0	0.3333	0	1

4 Conclusion

The advantage of this method to others is that a unique optimal solution is obtained, and it is revealed that DMU_j is not efficient.

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