On a Conjecture Concerning the Inverse Eigenvalue Problem of $4 \times 4$ Symmetric Doubly Stochastic Matrices

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Abstract

In this note, we prove that the conjecture giving in [14] concerning the inverse eigenvalue problem for $4 \times 4$ symmetric doubly stochastic matrices, is wrong. In addition, a new subset of the region where the decreasingly ordered spectra of $4 \times 4$ symmetric doubly stochastic lie, is found, and an alternative conjecture concerning the same problem is given.

Mathematics Subject Classification: 15A12; 15A18; 15A42; 15A48

Keywords: doubly stochastic matrices, inverse eigenvalue problem

1 Introduction

Let $\mathbb{C}$ denote the field of complex numbers, and $\mathbb{R}$ denote the field of real numbers. An $n \times n$ matrix with real entries is nonnegative if all of its entries are nonnegative. A nonnegative matrix such that each of its row and column sums is equal to 1, is called doubly stochastic. There is a big interest in the theory of doubly stochastic matrices because it is particularly endowed with a rich collection of applications in other areas of mathematics such as graph theory and combinatorics, numerical analysis, probability and statistics, and also in other disciplines such as quantum mechanics, communication theory, economics, and operation research, etc.; a clean exposition of this topic can be found in [1, 2, 3, 5, 6, 9, 16]. Of particular interest for this theory and also

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$^3$Research supported in part by LIU grants 2007/02
for the theory of nonnegative matrices is the inverse eigenvalue problem that has a lot of applications in many fields (see \[4, 7\]).

The inverse eigenvalue problem for doubly stochastic matrices is essentially equivalent to finding the region $\Theta_n$ of $\mathbb{C}^n$ where the spectra of doubly stochastic matrices lie. A subproblem of this problem is the symmetric inverse eigenvalue problem which is in turn equivalent to describing the region $\Theta_n^s$ of $\mathbb{R}^n$ where the decreasingly ordered spectra of symmetric doubly stochastic matrices lie. More precisely, $\Theta_n^s = \{\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n; \text{ where } 1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq -1 \text{ and there exists a symmetric doubly-stochastic with spectrum } \lambda \}$. For more on this subject see \[10, 12, 13, 14, 15, 17, 18, 19\] and the references therein.

First we introduce some notations. Let $M^+_s(n)$ be the class of all $n \times n$ symmetric nonnegative matrices. The set of all $n \times n$ doubly-stochastic (resp. symmetric doubly stochastic) matrices is denoted by $\Delta_n$ (resp. $\Delta^s_n$). Let $I_n$ be the $n \times n$ identity matrix and $J_n$ the $n \times n$ matrix whose all entries are $\frac{1}{n}$ and denote by $K_n$ to be the $n \times n$ matrix whose diagonal entries are zeros and all whose off-diagonal entries are all equal to $\frac{1}{n-1}$. If $p_1, p_2, ..., p_n$ are any points of $\mathbb{R}^n$, then their convex hull will be denoted by $\text{Conv}(p_1, p_2, ..., p_n)$. The line-segment joining $p_i$ to $p_j$ will be denoted by $[p_i, p_j]$.

Next let $E : M^+_s(n) \to \mathbb{R}^n$ be the map defined by $E(X) = (1, \lambda_2, ..., \lambda_n)$ where $(1, \lambda_2, ..., \lambda_n)$ is the decreasingly ordered set of eigenvalues of the matrix $X$. It is easy to see that $E(I_n) = (1,1,\ldots,1)$. Moreover in \[12\] it has been proved the following:

**Lemma 1.1** $E(J_n) = (1,0,\ldots,0)$ and $E(K_n) = (1,-\frac{1}{n-1},\ldots,-\frac{1}{n-1})$.

Clearly by definition $E(\Delta^s_n) = \Theta_n^s$. Concerning $\Theta_n$ and $\Theta_n^s$, we have the following theorem for which the proof can be found in \[12\].

**Theorem 1.2** For $n \geq 4$, $\Theta_n^s$ and $\Theta_n$ are not convex.

Recall that $\Delta_n$ is a convex polytope of dimension $(n-1)^2$ where its vertices are the $n \times n$ permutation matrices (see \[6\]). While $\Delta^s_n$ is a convex polytope of dimension $\frac{1}{2}n(n-1)$, its vertices were determined by \[11\] (see also \[8\]). They proved that if $A$ is a vertex of $\Delta^s_n$, then $A = \frac{1}{2}(P + P^T)$ for some permutation matrix $P$, although not every $\frac{1}{2}(P + P^T)$ is a vertex.

Next, we introduce the following definition which is useful for our study.

**Definition 1.3** A set $\Gamma$ of $\mathbb{R}^n$ is said to be star-convex with respect to a point $p \in \mathbb{R}^n$ if the line from any point in the set to $p$ is also contained in $\Gamma$.

Finally, in \[12\] it has been proved the following:

**Theorem 1.4** $\Theta_n^s$ is star convex with respect to any point of the line-segment $E(I_nK_n) = E(I_n)E(K_n) = [(1,1,\ldots,1), (1,-\frac{1}{n-1},\ldots,-\frac{1}{n-1})]$. 
2 Inverse eigenvalue problem for $4 \times 4$ symmetric doubly stochastic matrices.

Now let

$$\Gamma = \{(1, x, y, z) \in \mathbb{R}^4 : 1 \geq x \geq y \geq z \geq -1 \text{ and } 1 + x + y + z \geq 0\}.$$ 

Then clearly $\Gamma$ is a convex polytope with vertices given by:

$$\{(1, 1, 1, 1), (1, 0, 0, -1), (1, -1/3, -1/3, -1/3), (1, 1, 1, -1), (1, 1, -1, -1)\},$$

and obviously $\Theta_4^*$ is contained in $\Gamma$. By Theorem 1.2, we know that $\Theta_4^*$ is strictly contained in $\Gamma$. In [14] (see also [15]), a geometric description of a region $E_f$ of $\Theta_4^*$ has been found and is the maximum known subregion of $\Theta_4^*$. In addition, $E_f$ can be defined algebraically as the set of real 4-tuples $(1, x, y, z)$ where $1 \geq x \geq y \geq z \geq -1$ and $(x, y, z)$ satisfies either one of the following systems:

\[
\begin{align*}
&x + y + z + 1 \geq 0 \\
&y - z - \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2} \leq 0 \quad (1) \\
&y - z - \sqrt{2x^2 + 5y^2 + 5z^2 - 2xy - 2xz + 10yz + 6y + 6z + 2} \leq 0,
\end{align*}
\]

or

\[
\begin{align*}
&x + y + z + 1 \geq 0 \\
&y - z - \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2} \leq 0 \\
&y - z - \sqrt{10x^2 + y^2 + z^2 - 6xy - 6xz + 8y + 6z - 8x + 2y + 2z + 2} \leq 0 \\
&y - z - \sqrt{10x^2 + y^2 + z^2 - 6xy - 6xz + 2yz - 8x + 2y + 2z + 2} \leq 0.
\end{align*}
\]

In addition, the following conjecture has been made.

**Conjecture 2.1** Let $1 \geq x \geq y \geq z \geq -1$. Then $(1, x, y, z)$ is the spectrum of a $4 \times 4$ symmetric doubly-stochastic matrix if and only if $(1, x, y, z)$ satisfies one of the following systems:

\[
\begin{align*}
&x + y + z + 1 \geq 0 \\
&y - z - \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2} \leq 0 \\
&y - z - \sqrt{2x^2 + 5y^2 + 5z^2 - 2xy - 2xz + 10yz + 6y + 6z + 2} \leq 0,
\end{align*}
\]

or

\[
\begin{align*}
&x + y + z + 1 \geq 0 \\
&y - z - \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2} \leq 0 \\
&y - z - \sqrt{10x^2 + y^2 + z^2 - 6xy - 6xz + 8y + 6z - 8x + 2y + 2z + 2} \leq 0 \\
&y - z - \sqrt{10x^2 + y^2 + z^2 - 6xy - 6xz + 2yz - 8x + 2y + 2z + 2} \leq 0.
\end{align*}
\]

That is $\Theta_4^* = E_f$. 

3 A counter example

In this section, we give a counter example for Conjecture 2.5 and we also describe a new region $R_f$ of $\Theta^*_4$ such that $E_f$ is contained strictly in $R_f$.

First let $M(x, y)$ be the real $4 \times 4$ matrix whose entries depend on $x$ and $y$ and which is defined by: 

$$M(x, y) = \begin{pmatrix} \frac{(1+y)(y+x)}{1+y+2y} & \frac{(1+y-2y)^2}{2(1+y+2y)} & \frac{1-x}{2} & 0 \\
\frac{1+y+2y}{2(1+y+2y)} & \frac{1+y(y+x)}{1+y+2y} & 0 & \frac{1-x}{2} \\
\frac{2}{1-x} & 0 & 1-x & \frac{1+x}{2} \\
0 & \frac{2}{1-x} & 1-x & 0 \end{pmatrix}.$$ 

Now let $\Pi$ be the region in the plane defined by the following system:

$$\begin{cases} 
-1 \leq x \leq 1 \\
x + y \geq 0 \\
1 + x^2 - 2y^2 \geq 0. 
\end{cases}$$

(3)

Then (using Maple for example) one can easily check the following lemma:

**Lemma 3.1** If $(x, y) \in \Pi$, then $M(x, y)$ is a symmetric doubly stochastic matrix. In addition, the spectrum of $M(x, y)$ is given by $\{1, x, y, -\frac{x^2+y^2+xy+1}{1+x+2y}\}$.

As a straight forward conclusion is the following theorem.

**Theorem 3.2** For all $(x, y) \in \Pi$ with $x \geq y$, the image under the map $E$ of the matrix $M(x, y)$ is given by $E(M(x, y)) = (1, x, y, -\frac{x^2+y^2+xy+1}{1+x+2y})$ and clearly is contained in $\Theta^*_4$.

Next we give a counter example for Conjecture 2.5. Let $x = \frac{1}{2}$ and $y = \frac{1}{2}$, then $M(\frac{1}{2}, \frac{1}{2}) = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} & \frac{1}{4} & 0 \\
\frac{3}{2} & \frac{3}{2} & 0 & \frac{1}{4} \\
\frac{1}{4} & 0 & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}$ and $E(M(\frac{1}{2}, \frac{1}{2})) = (1, \frac{1}{2}, \frac{1}{2}, -\frac{4}{5})$ which can be easily checked that it does not satisfy neither of the two systems (1) and (2). Thus $E_f$ is strictly contained in the region $\Theta^*_4$, and we therefore obtained a new region $R_f$ of $\Theta^*_4$. We can describe geometrically $R_f$ as follows. Let $S$ be the surface in $\Gamma$ defined by:

$$S = \left\{(1, x, y, z) \in \Gamma : z = -\frac{x^2+xy+y+1}{1+x+2y}, \text{ and } x, y \in \Pi \right\}.$$ 

First, to describe the boundary of $S$ in $\Gamma$, let $\delta$ be the curve defined as the set of points in $\Gamma$ that are of the form 

$$(1, x, x, -\frac{2x^2+x+1}{1+3x})$$
in the plane $x = y$. Then an inspection shows that the curve $\delta$ is a hyperbola for which in $\Gamma$ it starts at the point $(1, 1, 1, -1)$ and ends at $(1, 0, 0, -1)$. Moreover, the surface $S$ can be obtained by joining via straight lines the point $(1, 1, 1, -1)$ to each point of $\delta$. Now by Theorem 1.4, the star-convexity property ensures that the region $R_f$ obtained by joining the line-segment $[(1, 1, 1, 1), (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})]$ to each point of the set $S \cup \text{Conv}((1, 1, -1, -1); (1, 0, 0, -1); (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}))$ is contained in $\Theta_s^4$. Thus we have the following theorem.

**Theorem 3.3** Let $1 \geq x \geq y \geq z \geq -1$. If $(1, x, y, z)$ satisfies the following system:

$$\begin{cases}
\quad x + y + z + 1 \geq 0 \\
\quad z + \frac{x^2 + xy + y + 1}{1 + x + 2y} \geq 0
\end{cases}$$

then $(1, x, y, z)$ is the spectrum of a $4 \times 4$ symmetric doubly stochastic matrix.

Now note that if $(1, x, y, z)$ satisfies either one of the systems (1) or (2), then $y - z \leq \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2}$. Squaring both sides and rearranging the terms, we obtain $z(1 + x + 2y + x^2 + xy + y + 1 \geq 0$. Since for $x, y \in \Pi$ with $x \geq y$ we have $1 + x + 2y \geq 0$, then we obtain $z + \frac{x^2 + xy + y + 1}{1 + x + 2y} \geq 0$. So that the region $E_f$ is contained in the region $R_f$. Moreover, the above counter example shows that $E_f$ is strictly contained in $R_f$. So that it is natural to conjecture the following.

**Conjecture 3.4** Let $1 \geq x \geq y \geq z \geq -1$. Then $(1, x, y, z)$ is the spectrum of a $4 \times 4$ symmetric doubly-stochastic matrix if and only if $(1, x, y, z)$ satisfies the following system

$$\begin{cases}
\quad x + y + z + 1 \geq 0 \\
\quad z + \frac{x^2 + xy + y + 1}{1 + x + 2y} \geq 0
\end{cases}$$

That is $\Theta_s^4 = R_f$.

**References**


Received: January 8, 2008