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On a Conjecture Concerning the Inverse Eigenvalue Problem of 4×4 Symmetric Doubly Stochastic Matrices

Issam Kaddoura[1] **and Bassam Mourad**[23]

Department of Mathematics, Lebanese International University
Beirut Campus, Al-Mouseitbeh, P.O. Box 14-6404, Beirut, Lebanon

Abstract

In this note, we prove that the conjecture giving in [14] concerning the inverse eigenvalue problem for 4×4 symmetric doubly stochastic matrices, is wrong. In addition, a new subset of the region where the decreasingly ordered spectra of 4×4 symmetric doubly stochastic lie, is found, and an alternative conjecture concerning the same problem is given.

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1 Introduction

Let **C** denote the field of complex numbers, and **R** denote the field of real numbers. An $n \times n$ matrix with real entries is *nonnegative* if all of its entries are nonnegative. A nonnegative matrix such that each of its row and column sums is equal to 1, is called *doubly stochastic*. There is a big interest in the theory of doubly stochastic matrices because it is particularly endowed with a rich collection of applications in other areas of mathematics such as graph theory and combinatorics, numerical analysis, probability and statistics, and also in other disciplines such as quantum mechanics, communication theory, economics, and operation research, etc.; a clean exposition of this topic can be found in [1, 2, 3, 5, 6, 9, 16]. Of particular interest for this theory and also

[1]corresponding author. Email: issam.kaddoura@liu.edu.lb
[2]bassam.mourad@liu.edu.lb
[3]Research supported in part by LIU grants 2007/02

for the theory of nonnegative matrices is the *inverse eigenvalue problem* that has a lot of applications in many fields (see [4, 7]).

The inverse eigenvalue problem for doubly stochastic matrices is essentially equivalent to finding the region Θ_n of $\mathbf{C}^n$ where the spectra of doubly stochastic matrices lie. A subproblem of this problem is the *symmetric inverse eigenvalue problem* which is in turn equivalent to describing the region Θ_n^s of $\mathbf{R}^n$ where the decreasingly ordered spectra of symmetric doubly stochastic matrices lie. More precisely, $\Theta_n^s = \{\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbf{R}^n$; where $1 = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$ and there exists a symmetric doubly-stochastic with spectrum λ $\}$. For more on this subject see [10, 12, 13, 14, 15, 17, 18, 19] and the references therein.

First we introduce some notations. Let $M_s^+(n)$ be the class of all $n \times n$ symmetric nonnegative matrices. The set of all $n \times n$ doubly-stochastic (resp. symmetric doubly stochastic) matrices is denoted by Δ_n (resp. Δ_n^s). Let I_n be the $n \times n$ identity matrix and J_n the $n \times n$ matrix whose all entries are $\frac{1}{n}$ and denote by K_n to be the $n \times n$ matrix whose diagonal entries are zeros and all whose off-diagonal entries are all equal to $\frac{1}{n-1}$. If $p_1,\ p_2, ...,\ p_n$ are any points of $\mathbf{R}^n$, then their convex hull will be denoted by $Conv(p_1, p_2, ..., p_n)$. The line-segment joining p_i to p_j will be denoted by $[p_i, p_j]$.

Next let $E : M_s^+(n) \to \mathbf{R}^n$ be the map defined by $E(X) =(1, \lambda_2, ..., \lambda_n)$ where $(1, \lambda_2, ..., \lambda_n)$ is the decreasingly ordered set of eigenvalues of the matrix X. It is easy to see that $E(I_n) = (1, 1, , ..., 1)$. Moreover in [12] it has been proved the following:

Lemma 1.1 $E(J_n) = (1, 0, ..., 0)$ *and* $E(K_n) = (1, -\frac{1}{n-1}, ..., -\frac{1}{n-1})$.

Clearly by definition $E(\Delta_n^s) = \Theta_n^s$. Concerning Θ_n and Θ_n^s, we have the following theorem for which the proof can be found in [12].

Theorem 1.2 *For* $n \geq 4$, Θ_n^s *and* Θ_n *are not convex.*

Recall that Δ_n is a convex polytope of dimension $(n-1)^2$ where its vertices are the $n \times n$ permutation matrices (see [6]). While Δ_n^s is a convex polytope of dimension $\frac{1}{2}n(n-1)$, its vertices were determined by [11] (see also [8]). They proved that if A is a vertex of Δ_n^s, then $A = \frac{1}{2}(P + P^T)$ for some permutation matrix P, although not every $\frac{1}{2}(P + P^T)$ is a vertex.

Next, we introduce the following definition which is useful for our study.

Definition 1.3 *A set* Γ *of* $\mathbf{R}^n$ *is said to be star-convex with respect to a point* $p \in \mathbf{R}^n$ *if the line from any point in the set to* p *is also contained in* Γ.

Finally, in [12] it has been proved the following:

Theorem 1.4 Θ_n^s *is star convex with respect to any point of the line-segment* $E(I_nK_n) = E(I_n)E(K_n) = [(1, 1, ..., 1), (1, -\frac{1}{n-1}, ..., -\frac{1}{n-1})]$.

2 Inverse eigenvalue problem for 4×4 symmetric doubly stochastic matrices.

Now let

$$\Gamma = \{(1, x, y, z) \in \mathbf{R}^4 : 1 \geq x \geq y \geq z \geq -1 \text{ and } 1 + x + y + z \geq 0\}.$$

Then clearly Γ is a convex polytope with vertices given by:

$$\{(1, 1, 1, 1), (1, 0, 0, -1), (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}), (1, 1, 1, -1), (1, 1, -1, -1)\},$$

and obviously Θ_4^s is contained in Γ. By Theorem 1.2, we know that Θ_4^s is strictly contained in Γ. In [14] (see also [15]), a geometric description of a region E_f of Θ_4^s has been found and is the maximum known subregion of Θ_4^s. In addition, E_f can be defined algebraically as the set of real 4-tuples $(1, x, y, z)$ where $1 \geq x \geq y \geq z \geq -1$ and (x, y, z) satisfies either one of the following systems:

$$\left\{ \begin{array}{l} x + y + z + 1 \geq 0 \\ y - z - \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2} \leq 0 \\ y - z - \sqrt{2x^2 + 5y^2 + 5z^2 - 2xy - 2xz + 10yz + 6y + 6z + 2} \leq 0, \end{array} \right. \quad (1)$$

or

$$\left\{ \begin{array}{l} x + y + z + 1 \geq 0 \\ y - z - \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2} \leq 0 \\ y - z - \sqrt{10x^2 + y^2 + z^2 - 6xy - 6xz + 2yz - 8x + 2y + 2z + 2} \leq 0. \end{array} \right. \quad (2)$$

In addition, the following conjecture has been made.

Conjecture 2.1 *Let $1 \geq x \geq y \geq z \geq -1$. Then $(1, x, y, z)$ is the spectrum of a 4×4 symmetric doubly-stochastic matrix if and only if $(1, x, y, z)$ satisfies one of the following systems:*

$$\left\{ \begin{array}{l} x + y + z + 1 \geq 0 \\ y - z - \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2} \leq 0 \\ y - z - \sqrt{2x^2 + 5y^2 + 5z^2 - 2xy - 2xz + 10yz + 6y + 6z + 2} \leq 0, \end{array} \right.$$

or

$$\left\{ \begin{array}{l} x + y + z + 1 \geq 0 \\ y - z - \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2} \leq 0 \\ y - z - \sqrt{10x^2 + y^2 + z^2 - 6xy - 6xz + 2yz - 8x + 2y + 2z + 2} \leq 0. \end{array} \right.$$

That is $\Theta_4^s = E_f$.

3 A counter example

In this section, we give a counter example for Conjecture 2.5 and we also describe a new region R_f of Θ_4^s such that E_f is contained strictly in R_f.

First let $M(x,y)$ be the real 4×4 matrix whose entries depend on x and y and which is defined by: $M(x,y) = \begin{pmatrix} \frac{(1+y)(y+x)}{1+x+2y} & \frac{(1+x^2-2y^2)}{2(1+x+2y)} & \frac{1-x}{2} & 0 \\ \frac{(1+x^2-2y^2)}{2(1+x+2y)} & \frac{(1+y)(y+x)}{1+x+2y} & 0 & \frac{1-x}{2} \\ \frac{1-x}{2} & 0 & 0 & \frac{1+x}{2} \\ 0 & \frac{1-x}{2} & \frac{1+x}{2} & 0 \end{pmatrix}$.

Now let Π be the region in the plane defined by the following system:

$$\begin{cases} -1 \leq x \leq 1 \\ x+y \geq 0 \\ 1+x^2-2y^2 \geq 0. \end{cases} \qquad (3)$$

Then (using Maple for example) one can easily check the following lemma:

Lemma 3.1 *If $(x,y)\in\Pi$, then $M(x,y)$ is a symmetric doubly stochastic matrix. In addition, the spectrum of $M(x,y)$ is given by $\left\{1, x, y, -\frac{x^2+xy+y+1}{1+x+2y}\right\}$.*

As a straight forward conclusion is the following theorem.

Theorem 3.2 *For all $(x,y)\in\Pi$ with $x\geq y$, the image under the map E of the matrix $M(x,y)$ is given by $E(M(x,y)) = \left(1, x, y, -\frac{x^2+xy+y+1}{1+x+2y}\right)$ and clearly is contained in Θ_4^s.*

Next we give a counter example for Conjecture 2.5. Let $x=\frac{1}{2}$ and $y=\frac{1}{2}$, then $M(\frac{1}{2},\frac{1}{2}) = \begin{pmatrix} \frac{3}{5} & \frac{3}{20} & \frac{1}{4} & 0 \\ \frac{3}{20} & \frac{3}{5} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}$ and $E(M(\frac{1}{2},\frac{1}{2})) = (1, \frac{1}{2}, \frac{1}{2}, \frac{-4}{5})$ which can be easily checked that it does not satisfy neither of the two systems (1) and (2). Thus E_f is strictly contained in the region Θ_4^s, and we therefore obtained a new region R_f of Θ_4^s. We can describe geometrically R_f as follows. Let S be the surface in Γ defined by:

$$S = \left\{(1,x,y,z)\in\Gamma : z = -\frac{x^2+xy+y+1}{1+x+2y}, \text{and } x,y\in\Pi\right\}.$$

First, to describe the boundary of S in Γ, let δ be the curve defined as the set of points in Γ that are of the form

$$(1,x,x,-\frac{2x^2+x+1}{1+3x})$$

in the plane $x = y$. Then an inspection shows that the curve δ is a hyperbola for which in Γ it starts at the point $(1, 1, 1, -1)$ and ends at $(1, 0, 0, -1)$. Moreover, the surface S can be obtained by joining via straight lines the point $(1, 1, -1, -1)$ to each point of δ. Now by Theorem 1.4, the star-convexity property ensures that the region R_f obtained by joining the line-segment $[(1, 1, 1, 1), (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})]$ to each point of the set

$$S \bigcup Conv((1, 1, -1, -1); (1, 0, 0, -1); (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}))$$

is contained in Θ_4^s. Thus we have the following theorem.

Theorem 3.3 *Let* $1 \geq x \geq y \geq z \geq -1$. *If* $(1, x, y, z)$ *satisfies the following system:*

$$\begin{cases} x + y + z + 1 \geq 0 \\ z + \frac{x^2+xy+y+1}{1+x+2y} \geq 0 \end{cases} \tag{4}$$

then $(1, x, y, z)$ *is the spectrum of a* 4×4 *symmetric doubly stochastic matrix.*

Now note that if $(1, x, y, z)$ satisfies either one of the systems (1) or (2), then $y - z \leq \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2}$. Squaring both sides and rearranging the terms, we obtain $z(1+x+2y)+x^2+xy+y+1 \geq 0$. Since for $x, y \in \Pi$ with $x \geq y$ we have $1+x+2y \geq 0$, then we obtain $z+\frac{x^2+xy+y+1}{1+x+2y} \geq 0$. So that the region E_f is contained in the region R_f. Moreover, the above counter example shows that E_f is strictly contained in R_f. So that it is natural to conjecture the following.

Conjecture 3.4 *Let* $1 \geq x \geq y \geq z \geq -1$. *Then* $(1, x, y, z)$ *is the spectrum of a* 4×4 *symmetric doubly-stochastic matrix if and only if* $(1, x, y, z)$ *satisfies the following system*

$$\begin{cases} x + y + z + 1 \geq 0 \\ z + \frac{x^2+xy+y+1}{1+x+2y} \geq 0. \end{cases}$$

That is $\Theta_4^s = R_f$.

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