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Matrices over Uniserial Ring

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Abstract

It is known that the ring of matrices over simple ring is simple. This is clarified by using the fact that two sided ideals of matrices over a ring R is matrices over two sided ideal of R. In this paper we study the right ideal of matrices over a ring R, and then using the result to prove that the ring of matrices over a uniserial ring is a serial ring.

The first Theorem will be the form of the right ideal of matrices over a ring *R*.

Theorem 1. *Let R be a ring and I a right ideal in* $M_n(R)$. *Then I =* $(U\ldots U)$ *for a right submodule U of* R^n.

Proof:

Let's look at the following set

$$U = \left\{ \begin{pmatrix} a_{11} \\ . \\ . \\ a_{n1} \end{pmatrix} \middle| \, (a_{ij}) \in I \right\}$$

It can be easily proved that U is a right submodule of R^n over R. Now it will be proved that $I = (U \dots U)$.

Let $M = (m_{ij}) \in I$. Because I is a right ideal, then

$$\begin{pmatrix} m_{1k} & 0 & . & 0 \\ . & . & & . \\ . & . & & . \\ m_{nk} & 0 & . & 0 \end{pmatrix} = ME_{k1} \in I .$$

Then

$$\begin{pmatrix} m_{1k} \\ . \\ . \\ m_{nk} \end{pmatrix} \in U .$$

So every column of M is contained in U. We have

$$I \subseteq (U....U)$$

Now let $M = (m_{ij}) \in (U...U)$. It will be proved that

$$\begin{pmatrix} 0 & . & m_{1k} & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & . & m_{nk} & 0 \end{pmatrix} \in I \ \forall \ k = 1,....,n$$

Because of $\begin{pmatrix} m_{1k} \\ . \\ . \\ m_{nk} \end{pmatrix} \in U$, there is a matrix $A = (a_{ij}) \in I$ with $\begin{pmatrix} m_{ik} \\ . \\ . \\ m_{nk} \end{pmatrix} = \begin{pmatrix} a_{11} \\ . \\ . \\ a_{n1} \end{pmatrix}$.

Because of *I* is a right ideal, then $\begin{pmatrix} 0 & . & m_{1k} & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & . & m_{nk} & 0 \end{pmatrix} = AE_{ik} \in I$. So $M \in I$. So we have $I \supseteq (U....U)$. We get $I = (U....U)$ ■

Next we will use Theorem 1 to see the structure of matrices over a uniserial ring. But we will look at two Lemmas before.

Lemma 2. *Let M be a right module over an Artinian ring R. Then M is uniserial if and only if the following chain is a composition series.*

$$0 \subseteq MJ^n \subseteq .... \subseteq MJ \subseteq M.$$

Proof.

$\Rightarrow$

Let *M/JM* be a module over *R/J*. By the way we have *R/J* is semi simple because *R* is Artinian. Then *M/JM* is semi simple. So $M / JM = \oplus M_i / JM$ with M_i / JM is simple modules. Then we can make two different composition series as follow :

$$JM \subseteq M_1 \subseteq M_1 \oplus M_2 \subseteq \cdots \subseteq M$$

and

$$JM \subseteq M_2 \subseteq M_1 \oplus M_2 \subseteq \cdots \subseteq M.$$

This is impossible because *M* is uniserial. So *M/JM* must be simple. Next we also have JM / J^2M simple.

$\Leftarrow$

Let N_1 be a maximal right submodule of *M*. If $MJ \neq N_1$, then $MJ + N_1 = M$. Following Nakayama [2], because *R* is Artinian, then *MJ* << *M*. So $M = N_1$. This is contradict the fact that N_1 is a maximal right submodule. So it must be $MJ = N_1$. The process is continued so that we have $MJ^2 = N_2$ and so forth. So we have that *M* is uniserial. ■

From Lemma 2, it can be seen that a ring *R* is uniserial if and only if the chain $0 \subseteq J^n \subseteq \cdots \subseteq J \subseteq R$ is a composition series in the mean that *J* is a maximal right and left ideal.

Lemma 3. *Let R be a ring. Then I is a right submodule of*

$$i^{th}\ line \cdots \begin{pmatrix} 0 \\ . \\ R \\ . \\ 0 \end{pmatrix} \subseteq R^n \Leftrightarrow I = \begin{pmatrix} 0 \\ . \\ J_i \\ . \\ 0 \end{pmatrix} \text{ with } J_i \text{ is a right ideal of } R$$

Proof :

$\Leftarrow$

Let $\begin{pmatrix} 0 \\ . \\ a_i \\ . \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ . \\ b_i \\ \\ \end{pmatrix} \in I; r \in R$. Because of J_i is a right ideal of *R*, then

$$\begin{pmatrix} 0 \\ . \\ a_i \\ . \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ . \\ b_i \\ \\ \end{pmatrix} \in I; r \in R \text{ and } \begin{pmatrix} 0 \\ . \\ a_i \\ . \\ 0 \end{pmatrix} r \in I .$$

So I is a right R-submodule of $\begin{pmatrix} 0 \\ . \\ R \\ . \\ 0 \end{pmatrix}$.

$\Rightarrow$

We write $J_i = \left\{ a_i \in R \,\middle|\, \exists \begin{pmatrix} 0 \\ . \\ a_i \\ . \\ 0 \end{pmatrix} \in I \right\}$. It is important that a_i is in the i^{th} line. It can be easily seen that J_i is a right ideal in R. Let $\begin{pmatrix} 0 \\ . \\ a_i \\ . \\ 0 \end{pmatrix} \in I$. It means that $a_i \in J_i$. So $I \subseteq \begin{pmatrix} 0 \\ . \\ J_i \\ . \\ 0 \end{pmatrix}$.

Now let $\begin{pmatrix} 0 \\ . \\ a_i \\ . \\ 0 \end{pmatrix} \in \begin{pmatrix} 0 \\ . \\ J_i \\ . \\ 0 \end{pmatrix}$. So $\begin{pmatrix} 0 \\ . \\ a_i \\ . \\ 0 \end{pmatrix} \in I$. And so $I \supseteq \begin{pmatrix} 0 \\ . \\ J_i \\ . \\ 0 \end{pmatrix}$. We have $I = \begin{pmatrix} 0 \\ . \\ J_i \\ . \\ 0 \end{pmatrix}$. ■

Theorem 4. *Let R is a uniserial ring. Then $M_n(R)$ is a serial ring.*

Proof:

R is a serial ring, then the following composition series is unique

$$0 \subseteq J^k \subseteq \cdots \subseteq J \subseteq R .$$

We can see that

$$M_n(R) = \begin{pmatrix} R & . & . & R \\ 0 & & & 0 \\ . & & & . \\ 0 & . & . & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & . & . & 0 \\ 0 & & & 0 \\ . & & & . \\ R & . & . & R \end{pmatrix} .$$

Let see the right ideal $\begin{pmatrix} 0 & . & . & 0 \\ . & & & 0 \\ R & & & R \\ 0 & . & . & 0 \end{pmatrix}$. Because the right ideal in R is only J^t for some t,

and from Theorem 1 for the form of right ideal in $M_n(R)$, and the right submodule of

$i^{th}\ line \cdots \begin{pmatrix} 0 \\ . \\ R \\ . \\ 0 \end{pmatrix} \subseteq R^n$ is as in Lemma 3, then the right ideal

$\begin{pmatrix} 0 & . & . & 0 \\ . & & & 0 \\ R & & & R \\ 0 & . & . & 0 \end{pmatrix}$ is a uniserial right module with composition series as follow

$$\begin{pmatrix} 0 & . & . & 0 \\ . & & & 0 \\ 0 & & & 0 \\ 0 & . & . & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & . & . & 0 \\ . & & & 0 \\ J^k & & & J^k \\ 0 & . & . & 0 \end{pmatrix} \subseteq \cdots \subseteq \begin{pmatrix} 0 & . & . & 0 \\ . & & & 0 \\ J & & & J \\ 0 & . & . & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & . & . & 0 \\ . & & & 0 \\ R & & & R \\ 0 & . & . & 0 \end{pmatrix} .$$

So $M_n(R)$ is a direct sum of right uniserial modules. So $M_n(R)$ is a serial ring.■

References

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2. D.S.Passman, A Course in Ring Theory, 1991.

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