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Hypercyclic and Compact Composition Operators on Banach Spaces of Formal Power Series

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"Dedicated to Mola Ali"

Abstract

In this paper we consider the composition operator C_φ acting between the weighted Hardy spaces $H^p(\beta_1)$ and $H^q(\beta_2)$, for $1 < q \leq p < \infty$. We investigate the hypercyclicity of the composition operator C_φ acting between weighted Hardy spaces in the unit disk. Furthermore, the characterization of the fixed points of the composition operator C_φ will be considered.

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1 Introduction

Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 \leq p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that $||f||^p = ||f||_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty$. The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ shall be used whether or not the series converges for any value of z. Let $H^p(\beta)$ denotes the space of such formal power series. These are reflexive Banach spaces with the norm $||.||_\beta$ and the dual of $H^p(\beta)$ is $H^q(\beta^{p/q})$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta^{p/q} = \{\beta(n)^{p/q}\}_n$. Also if $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n \in H^q(\beta^{p/q})$, then $||g||^q = \sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta(n)^p$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ for all nonnegative integer k and then $\{f_k\}$ is a basis such that $\|f_k\| = \beta(k)$. If $\lim_n \frac{\beta(n+1)}{\beta(n)} = 1$, then $H^p(\beta)$ consists of functions analytic on the open unit disc U. It is convenient and helpful to introduce the notation $< f, g >$ to stand for $g(f)$ where $f \in H^p(\beta)$ and $g \in H^p(\beta)^*$. Note that $< f, g >= \sum_{n=0}^{\infty} \hat{f}(n) \; \overline{\hat{g}(n)} \beta(n)^p$. The spaces $H^p(\beta)$ are also called as weighted Hardy spaces.

We denote the set of multipliers

$$\{\varphi \in H^p(\beta) : \varphi H^p(\beta) \subseteq H^p(\beta)\}$$

by $H^p_\infty(\beta)$ and the operator of multiplication by φ on $H^p(\beta)$ by M_φ with $\|\varphi\|_\infty = \|M_\varphi\|$.

A complex number λ is said to be a bounded point evaluation on $H^p(\beta)$ if the functional of point evaluation at λ, e_λ, is bounded. A complex number λ is a bounded point evaluation on $H^p(\beta)$ if and only if $\left\{ \frac{\lambda^n}{\beta(n)} \right\}_n \in l^q$ ([6]). If $\liminf \beta(n)^{\frac{1}{n}} = 1$ then for each λ in the open unit disk, the functional of evaluation at λ, e_λ, is a bounded linear functional and we have

$$e_\lambda(z) = \sum_{n=0}^{\infty} \frac{\bar{\lambda}^n z^n}{\beta(n)^p},$$

and

$$\|e_\lambda\|^q = \sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q}.$$

The function φ in $H^p(\beta)$ that maps the unit disc U into itself induces a composition operator C_φ on $H^p(\beta)$ defined by $C_\varphi f = f \circ \varphi$.

Let X be a complex Banach space and $B(X)$ be the set of bounded linear operators from X into itself. If $T \in B(X)$, then the orbit of a vector $x \in X$ is the set $Orb(T, x) = \{T^n x : n \in \mathbf{N} \cup \{\mathbf{0}\}\}$. The operator T is called hypercyclic if $Orb(T, x)$ is dense in X for some $x \in X$.

Some sources on these topics are [1–15].

2 Main Results

We study the hypercyclicity of the composition operator acting between the weighted Hardy spaces $H^p(\beta_1)$ and $H^q(\beta_2)$ where $1 < q \leq p < \infty$. For similar discussion on the Hardy space H^2 see [2]. Also, Furthermore, the characterization of the fixed points of a compact composition operator C_φ will be considered.

Theorem 1. Suppose that φ is an analytic self-map of the open unit disc and the composition operator $C_\varphi : H^p(\beta_1) \to H^q(\beta_2)$ is bounded where $1 < q \leq p < \infty$ and $\beta_1(n) \leq \beta_2(n)$ for all n. Also, let $\frac{1}{p} + \frac{1}{p'} = 1$ and $\sum_{n=0}^{\infty} \frac{1}{\beta_1(n)^{p'}} < \infty$. If C_φ is compact, then φ has exactly one fixed point in $\bar{U}$.

Proof. First note if

$$g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n \in H^q(\beta_2),$$

then by the Hölder inequality we have

$$\begin{aligned} |g(z)| &= |\sum_{n=0}^{\infty} \hat{g}(n) z^n| = |\sum_{n=0}^{\infty} (\hat{g}(n)\beta_2(n)) \frac{z^n}{\beta_2(n)}| \\ &\leq (\sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta_2(n)^q)^{1/q} . (\sum_{n=0}^{\infty} \frac{|z|^{nq'}}{\beta_2(n)^{q'}})^{1/q'} \\ &\leq ||g||_{H^q(\beta_2)} (\sum_{n=0}^{\infty} \frac{1}{\beta_2(n)^{q'}})^{1/q'} \end{aligned}$$

for all z in U where $\frac{1}{q} + \frac{1}{q'} = 1$. Hence for all g in $H^q(\beta_2)$,

$$||g||_U \leq d||g||_{H^q(\beta_2)}$$

where

$$d = (\sum_{n=0}^{\infty} \frac{1}{\beta_1(n)^{q'}})^{1/q'}.$$

Since $\sum_{n=0}^{\infty} \frac{1}{\beta_1(n)^{p'}} < \infty$, each point of $\bar{U}$ is a bounded point evaluation for both spaces $H^p(\beta_1)$ and $H^q(\beta_2)$. The functional of evaluation at ω ($w \in \bar{U}$) on both spaces $H^p(\beta_1)$ and $H^q(\beta_2)$ is denoted respectively by $e_w^{(p)}$ and $e_w^{(q)}$. Let λ be the Denjoy-Wolff point of φ. Put $M_{p,\lambda} = ker\ e_\lambda^{(p)}$ and $M_{q,\lambda} = ker\ e_\lambda^{(q)}$. Clearly $M_{p,\lambda}$ and $M_{q,\lambda}$ are closed subspaces of $H^p(\beta_1)$ and $H^q(\beta_2)$ respectively and also $C_\varphi M_{p,\lambda} \subseteq M_{q,\lambda}$. Now let $T : M_{p,\lambda} \to M_{q,\lambda}$ be the restriction of C_φ to $M_{p,\lambda}$. Then T is compact and so there exists α in the point spectrum of T, such that the spectral radius of T, $r(T)$, is equal to $|\alpha|$. Hence there exists a nonzero function f in $M_{p,\lambda} \cap M_{q,\lambda}$ such that $Tf = \alpha f$. Let $z_0 \in U$ be such that $f(z_0) \neq 0$. Then we have

$$\alpha^n f(z_0) = (T^n f)(z_0) = f(\varphi_n(z_0))$$

for all $n \in \mathbb{N}$ (here φ_n is the nth iterate of φ). Letting $n \to \infty$, we get $\alpha^n f(z_0) \to 0$ since $\varphi_n(z_0) \to \lambda$ and $f \in M_{p,\lambda}$. Thus it should be $|\alpha| < 1$ which implies that $r(T) < 1$. But $r(T) = \lim_n ||T^n||^{1/n}$, hence $||T^n|| \to 0$. Note that $f_1 - \lambda f_0 \in M_{p,\lambda}$ and

$$||\varphi_n - \lambda||_{H^q(\beta_2)} = ||T^n(f_1 - \lambda f_0)|| \to 0$$

as $n \to \infty$. But as we saw earlier, we have

$$||\varphi_n - \lambda||_U \leq d||\varphi_n - \lambda||_{H^q(\beta_2)}$$

where

$$d = (\sum_{n=0}^{\infty} \frac{1}{\beta_2(n)^{q'}})^{1/q'}$$

and $\frac{1}{q} + \frac{1}{q'} = 1$. Hence $\varphi_n(z) \to \lambda$ for all z in $\bar{U}$. Now if w is another fixed point of φ in $\bar{U}$, then we have $0 = \lim_n(\varphi_n(w) - \lambda) = w - \lambda$ and so $w = \lambda$. This completes the proof. □

Theorem 2. For all n, let $\beta_1(n) \leq \beta_2(n)$ and also suppose that for all $1 < q \leq p < \infty$, the composition operator $C_\psi : H^p(\beta_1) \to H^q(\beta_2)$ is bounded whenever ψ is an analytic self-map of the open unit disc with $||\psi||_U < 1$. If φ is an analytic self-map of the open unit disc and $||\varphi||_U < 1$, then $C_\varphi : H^p(\beta_1) \to H^q(\beta_2)$ is compact.

Proof. By the assumptions of the theorem C_φ is bounded from $H^p(\beta_1)$ into $H^q(\beta_2)$. Now let r_1 be such that $||\varphi||_U < r_1 < 1$ and put $\psi(z) = r_1 z$. Then $||\psi||_U = r_1 < 1$ and so $C_\psi : H^p(\beta_1) \to H^p(\beta_1)$ is bounded. Also, if $h = \frac{1}{r_1}\varphi$, then $||h||_U < 1$ and so $C_h : H^p(\beta_1) \to H^q(\beta_2)$ is bounded and we have $\varphi = \psi \circ h$. Hence $C_\varphi = C_h C_\psi$. Now it is sufficient to show that C_ψ is compact. For this note that $||f_n||_{H^p(\beta_1)} = \beta_1(n)$ where $f_n(z) = z^n$ for all n. Also, we have

$$C_\psi(\frac{f_n}{\beta_1(n)}) = \frac{f_n \circ \psi}{\beta_1(n)} = \frac{\psi^n}{\beta_1(n)} = r_1^n \frac{f_n}{\beta_1(n)}$$

for all n. From this we can conclude that C_ψ is indeed compact. This completes the proof. □

Corollary 3. Under the conditions of Theorem 2, C_φ is completely continuous.

Proof. Since every compact operator is completely continuous, the proof is complete. □

Theorem 4. Suppose that φ is an analytic self-map of U that fixes a point of $\bar{U}$ and $C_\varphi : H^p(\beta_1) \to H^q(\beta_2)$ is bounded where $1 < q \leq p < \infty$ and $\beta_1(n) \leq \beta_2(n)$ for all n. If $\frac{1}{p} + \frac{1}{p'} = 1$ and $\sum_{n=0}^{\infty} \frac{1}{\beta_1(n)^{p'}} < \infty$, then C_φ is not hypercyclic.

Proof. Note that by the assumptions of the theorem C_φ is bounded and each point of $\bar{U}$ is a bounded point evaluation for both spaces $H^p(\beta_1)$ and $H^q(\beta_2)$. The functional of evaluation at ω ($w \in \bar{U}$) on both spaces $H^p(\beta_1)$ and $H^q(\beta_2)$ is denoted respectively by $e_w^{(p)}$ and $e_w^{(q)}$. Now suppose $\alpha \in \bar{U}$ is a fixed point for φ. Then $< f, e_\alpha^{(p)} >= f(\alpha)$ for all f in $H^p(\beta_1)$. Fix $f \in H^p(\beta_1)$, to be regarded as a hypercyclic vector candidate.

If g belongs to the closure of $Orb(C_\varphi, f)$, then for some subsequence $n_k \to +\infty$ we have $C_{\varphi_{n_k}} f \to g$. Thus we have

$$\begin{aligned} g(\alpha) =< g, e_\alpha^{(q)} > &= \lim_k < C_{\varphi_{n_k}} f, e_\alpha^{(q)} >= \lim_k < f, C^*_{\varphi_{n_k}} e_\alpha^{(q)} > \\ &= \lim_k < f, e^{(p)}_{\varphi_{n_k}(\alpha)} >= \lim_k f(\varphi_{n_k}(\alpha)) = f(\alpha). \end{aligned}$$

This implies that no orbit can be dense in $H^q(\beta_2)$ and so $C_\varphi : H^p(\beta_1) \to H^q(\beta_2)$ is not hypercyclic, thus the proof is complete. □

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