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A Hardy-Hilbert Type Inequality

Namita Das

P. G. Department of Mathematics
Utkal University, Bhubaneswar
Orissa, India 751004
namitadas440@yahoo.co.in

Srinibas Sahoo

Department of Mathematics
Banki College, Banki
Orissa, India 754008
sahoosrinibas@yahoo.co.in

Abstract: By introducing two functions $u(x)$ and $v(x)$, we give a new Hardy-Hilbert type inequality, which is a generalization of Hardy-Hilbert's inequality. As applications, the equivalent form and it's discrete version inequality are derived.

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1. INTRODUCTION

Let $1/p + 1/q = 1 (p > 1), f, g \geq 0$ satisfy $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$. Then the well known Hardy-Hilbert's integral inequality (see [1]) is given by

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{sin(\pi/p)} \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}}; \tag{1.1}$$

and an equivalent form is given by

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx\right)^p dy < \left[\frac{\pi}{sin(\pi/p)}\right]^p \int_0^\infty f^p(x)dx; \tag{1.2}$$

where the constant factor $\pi/sin(\pi/p)$ and $[\pi/sin(\pi/p)]^p$ are the best possible. The corresponding double series inequality which is known as Hardy-Hilbert's inequality (see [1]) is : if $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n, b_n \geq 0$ satisfy $0 < \sum_{n=1}^\infty a_n^p < \infty$

and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$,then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{1/q}; \tag{1.3}$$

and an equivalent form is

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\sum_{n=1}^{\infty} a_n^p; \tag{1.4}$$

where the constant factor $\pi/\sin(\pi/p)$ and $[\pi/\sin(\pi/p)]^p$ are the best possible. Inequalities (1.1) and (1.3) are important in analysis and its applications (cf.Mitrinovic et al.[2]).Recently many generalization and refinements of these inequalities were also obtained.

Hardy et al.[1] gave an inequality, under the same condition of (1.1), similar to (1.1) as:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{\max\{x,y\}}dxdy < pq\left(\int_0^{\infty} f^p(x)dx\right)^{1/p}\left(\int_0^{\infty} g^q(x)dx\right)^{1/q}; \tag{1.5}$$

and an equivalent form is given by

$$\int_0^{\infty}\left(\int_0^{\infty}\frac{f(x)}{\max\{x,y\}}dx\right)^p dy < (pq)^p\int_0^{\infty} f^p(x)dx; \tag{1.6}$$

where the constant factor pq and $(pq)^p$ are the best possible. The corresponding double series inequality, under the same condition of (1.3), similar to (1.3) is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{max\{m,n\}} < pq\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{1/q}; \tag{1.7}$$

and an equivalent form is

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{max\{m,n\}}\right)^p < [pq]^p\sum_{n=1}^{\infty} a_n^p; \tag{1.8}$$

where the constant factor pq and $(pq)^p$ are the best possible. Recently Yongjin et al.[5] gave a common generalization of (1.1) and (1.5), for the case $p = q = 2$ as follows : if $f, g \geq 0, f, g \in L^2(0,\infty)$ then for $A > 0, B \geq 0$,

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{A\max\{x,y\}+B\max\{x,y\}}dxdy\\ &\qquad < D\,(A,B)\left(\int_0^{\infty} f^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^{\infty} g^2(x)dx\right)^{\frac{1}{2}};\end{aligned} \tag{1.9}$$

where the constant factor $D\,(A,B)$ defined by

$$D\,(A,B) = \int_0^{\infty}\frac{1}{A\max\{1,t\}+B\max\{1,t\}}t^{-\frac{1}{2}}dt$$

is the best possible.

In this paper, taking two different functions $u(x)$ and $v(x)$, we derive a new Hardy-Hilbert type inequality, which is an extension of (1.9) and a generalization of both (1.1) and (1.5). As applications, the equivalent form and it's discrete version inequality are considered.

2. SOME LEMMAS

First, We need the formula of the $\beta-$function as (cf.Wang et al.[3]):

$$B(p,q)=\int_0^\infty \frac{1}{(1+t)^{p+q}}t^{p-1}dt=B(q,p) \tag{2.1}$$

Lemma 2.1. *Let* $p>1, \frac{1}{p}+\frac{1}{q}=1$. *For* $A\geq 0, B>0$ *define*

$$K(A,B)=\int_0^1 \frac{1}{At+B}t^{-\frac{1}{q}}dt+\int_1^\infty \frac{1}{A+Bt}t^{-\frac{1}{q}}dt \tag{2.2}$$

Then

$$0<K(A,B)<\infty.$$

In particular for $A>0, B>0$,

$$K(A,B)=\frac{1}{A^{\frac{1}{q}}B^{\frac{1}{p}}}\int_0^{\frac{A}{B}} \frac{1}{1+t}t^{-\frac{1}{q}}dt+\frac{1}{A^{\frac{1}{p}}B^{\frac{1}{q}}}\int_{\frac{B}{A}}^\infty \frac{1}{1+t}t^{-\frac{1}{q}}dt \tag{2.3}$$

$$K(0,1)=pq \tag{2.4}$$

$$K(1,1)=\frac{\pi}{\sin\frac{\pi}{p}} \tag{2.5}$$

Proof. For $A>0, B>0$, setting $t=\frac{Bu}{A}$ in the first integral and $\frac{Au}{B}$ in the second integral of (2.2), we get (2.3). Hence by (2.1), we have

$$\begin{aligned}0<K(A,B)&<\frac{1}{A^{\frac{1}{q}}B^{\frac{1}{p}}}\int_0^\infty \frac{1}{1+t}t^{-\frac{1}{q}}dt+\frac{1}{A^{\frac{1}{p}}B^{\frac{1}{q}}}\int_0^\infty \frac{1}{1+t}t^{-\frac{1}{q}}dt\\&=\left(\frac{1}{A^{\frac{1}{q}}B^{\frac{1}{p}}}+\frac{1}{A^{\frac{1}{p}}B^{\frac{1}{q}}}\right)B\left(\frac{1}{p},\frac{1}{q}\right)\\&<\infty.\end{aligned}$$

For $A=0, B>0$, we have

$$0<K(0,B)=\frac{1}{B}\left(\int_0^1 t^{-\frac{1}{q}}dt+\int_1^\infty t^{-1-\frac{1}{q}}dt\right)=\frac{1}{B}(p+q)=\frac{pq}{B}<\infty.$$

(2.5) follows from (2.3) and (2.1). This completes the lemma. $\square$

Notation:- For $-\infty\leq a<b\leq\infty$, we denote $\mathcal{F}(a,b)$ as the class of functions $u:(a,b)\to(0,\infty)$ such that $u(x)$ is strictly increasing in (a,b) with $u(a+)=0$ and $u(b-)=\infty$.

Lemma 2.2. *Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1$. $u \in \mathcal{F}(a,b)$ and $v \in \mathcal{F}(c,d)$, define $\omega(u,v,q,x)$ and $\omega(v,u,p,y)$ as*

$$\omega(u,v,q,x) = \int_c^d \frac{(v(y))^{-\frac{1}{q}} v'(y)}{A\min\{u(x),v(y)\} + B\max\{u(x),v(y)\}} dy,\ x \in (a,b) \quad (2.6)$$

$$\omega(v,u,p,y) = \int_a^b \frac{(u(x))^{-\frac{1}{p}} u'(x)}{A\min\{u(x),v(y)\} + B\max\{u(x),v(y)\}} dx,\ y \in (c,d) \quad (2.7)$$

Then

$$\omega(u,v,q,x) = K(A,B)(u(x))^{-\frac{1}{q}} \quad (2.8)$$

and

$$\omega(v,u,p,y) = K(A,B)(v(y))^{-\frac{1}{p}}. \quad (2.9)$$

Proof. Setting $t = \frac{v(y)}{u(x)}$ in (2.6), we get

$$\begin{aligned} \omega(u,v,q,x) &= (u(x))^{-\frac{1}{q}} \int_0^\infty \frac{1}{A\min\{1,t\} + B\max\{1,t\}} t^{-\frac{1}{q}} dt \\ &= (u(x))^{-\frac{1}{q}} \left(\int_0^1 \frac{1}{At+B} t^{-\frac{1}{q}} dt + \int_1^\infty \frac{1}{A+Bt} t^{-\frac{1}{q}} dt \right) \\ &= K(A,B)(u(x))^{-\frac{1}{q}}. \end{aligned}$$

Setting $s = \frac{u(x)}{v(y)}$ in (2.7) and then setting $t = \frac{1}{s}$, we get

$$\begin{aligned} \omega(v,u,p,y) &= (v(y))^{-\frac{1}{p}} \int_0^\infty \frac{1}{A\min\{1,s\} + B\max\{1,s\}} s^{-\frac{1}{p}} dt \\ &= (v(y))^{-\frac{1}{p}} \left(\int_0^1 \frac{1}{As+B} s^{-\frac{1}{p}} ds + \int_1^\infty \frac{1}{A+Bs} s^{-\frac{1}{p}} ds \right) \\ &= (v(y))^{-\frac{1}{p}} \left(\int_1^\infty \frac{1}{A+Bt} t^{-\frac{1}{q}} dt + \int_0^1 \frac{1}{At+B} t^{-\frac{1}{q}} dt \right) \\ &= K(A,B)(v(y))^{-\frac{1}{p}}. \end{aligned}$$

This completes the lemma. □

Lemma 2.3. *Let the assumption of the lemma-2.2 holds. Take $a_1 = u^{-1}(1)$ and $c_1 = v^{-1}(1)$. Then for sufficiently small $\varepsilon > 0$,*

$$\begin{aligned} I := &\int_{a_1}^b \int_{c_1}^d \frac{(u(x))^{-\frac{1+\varepsilon}{p}} u'(x)(v(y))^{-\frac{1+\varepsilon}{q}} v'(y)}{A\min\{u(x),v(y)\} + B\max\{u(x),v(y)\}} dxdy \\ &> \frac{1}{\varepsilon}\{K(A,B) + \circ(1)\} - \bigcirc(1). \end{aligned} \quad (2.10)$$

Proof. For fixed $x \in (a,b)$, setting $t = \frac{v(y)}{u(x)}$, we have

$$\begin{aligned} I &= \int_{a_1}^{b} \frac{u'(x)}{(u(x))^{1+\varepsilon}} \left(\int_{\frac{1}{u(x)}}^{\infty} \frac{t^{-\frac{1+\varepsilon}{q}}}{A\min\{1,t\} + B\max\{1,t\}} dt \right) dx \\ &= \int_{a_1}^{b} \frac{u'(x)}{(u(x))^{1+\varepsilon}} dx \int_{0}^{\infty} \frac{t^{-\frac{1+\varepsilon}{q}}}{A\min\{1,t\} + B\max\{1,t\}} dt \\ &\quad - \int_{a_1}^{b} \frac{u'(x)}{(u(x))^{1+\varepsilon}} \left(\int_{0}^{\frac{1}{u(x)}} \frac{t^{-\frac{1+\varepsilon}{q}}}{A\min\{1,t\} + B\max\{1,t\}} dt \right) dx \\ &= I_1 - I_2 \quad \text{(say)} \end{aligned}$$

Then

$$\begin{aligned} I_1 &= \frac{1}{\varepsilon} \left\{ \int_{0}^{\infty} \frac{t^{-\frac{1}{q}}}{A\min\{1,t\} + B\max\{1,t\}} dt + \int_{0}^{\infty} \frac{t^{-\frac{1}{q}}(t^{-\frac{\varepsilon}{q}} - 1)}{A\min\{1,t\} + B\max\{1,t\}} dt \right\} \\ &= \frac{1}{\varepsilon} \{ K(A,B) + \circ(1) \} \ \text{ as } \ \varepsilon \to 0^{+}. \end{aligned}$$

Since $x \geq a_1$, so $u(x) \geq 1$. Hence

$$\begin{aligned} I_2 &= \int_{a_1}^{b} \frac{u'(x)}{(u(x))^{1+\varepsilon}} \left(\int_{0}^{\frac{1}{u(x)}} \frac{t^{-\frac{1+\varepsilon}{q}}}{At + B} dt \right) dx \\ &< \frac{1}{B} \int_{a_1}^{b} \frac{u'(x)}{u(x)} \left(\int_{0}^{\frac{1}{u(x)}} t^{-\frac{1+\varepsilon}{q}} dt \right) dx \\ &= \frac{1}{B} \left(\frac{1}{q} - \frac{\varepsilon}{q} \right)^{-2}. \end{aligned}$$

Hence (2.10) is valid.The lemma is proved. □

3. MAIN RESULTS

Theorem 3.1. *Let* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$. $u \in \mathcal{F}(a,b)$ *and* $v \in \mathcal{F}(c,d)$. *If* $f, g \geq 0$ *satisfy* $0 < \int_a^b f^p(x)dx < \infty$ *and* $0 < \int_c^d g^q(x)dx < \infty$ *then*

$$\begin{aligned} J := \int_{a}^{b} \int_{c}^{d} \frac{(u'(x))^{\frac{1}{q}} (v'(y))^{\frac{1}{p}} f(x) g(y)}{A\min\{u(x), v(y)\} + B\max\{u(x), v(y)\}} dx dy \\ < K(A,B) \left(\int_{a}^{b} f^{p}(x) dx \right)^{\frac{1}{p}} \left(\int_{c}^{d} g^{q}(x) dx \right)^{\frac{1}{q}}; \end{aligned} \tag{3.1}$$

where the constant factor $K(A,B)$ *is the best possible. In particular,*
(i) *for* $A=B=1$, *we have*

$$\begin{aligned}&\int_a^b\int_c^d \frac{(u'(x))^{\frac{1}{q}}(v'(y))^{\frac{1}{p}}f(x)g(y)}{u(x)+v(y)}dxdy\\&\qquad<\frac{\pi}{\sin\frac{\pi}{p}}\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(\int_c^d g^q(x)dx\right)^{\frac{1}{q}};\end{aligned}\tag{3.2}$$

(ii) *for* $A=0, B=1$, *we have*

$$\begin{aligned}&\int_a^b\int_c^d \frac{(u'(x))^{\frac{1}{q}}(v'(y))^{\frac{1}{p}}f(x)g(y)}{\max\{u(x),v(y)\}}dxdy\\&\qquad<pq\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(\int_c^d g^q(x)dx\right)^{\frac{1}{q}};\end{aligned}\tag{3.3}$$

where the constant factors $\frac{\pi}{\sin\frac{\pi}{p}}$ *and* pq *are the best possible.*

Proof. By the Hölder's inequality with weight (cf. Kuang [4]), we have

$$\begin{aligned}J=&\int_a^b\int_c^d\frac{1}{A\min\{u(x),v(y)\}+B\max\{u(x),v(y)\}}\\&\times\left[\left(\frac{u(x)}{v(y)}\right)^{\frac{1}{pq}}(v'(y))^{\frac{1}{p}}f(x)\right]\left[\left(\frac{v(y)}{u(x)}\right)^{\frac{1}{pq}}(u'(x))^{\frac{1}{q}}g(y)\right]dxdy\\ \le&\left\{\int_a^b\left(\int_c^d\frac{(v(y))^{-\frac{1}{q}}v'(y)}{A\min\{u(x),v(y)\}+B\max\{u(x),v(y)\}}dy\right)(u(x))^{\frac{1}{q}}f^p(x)dx\right\}^{\frac{1}{p}}\\&\times\left\{\int_c^d\left(\int_a^b\frac{(u(x))^{-\frac{1}{p}}u'(x)}{A\min\{u(x),v(y)\}+B\max\{u(x),v(y)\}}dx\right)(v(y))^{\frac{1}{p}}g^q(y)dy\right\}^{\frac{1}{q}}\end{aligned}\tag{3.4}$$

If (3.4) takes the form of equality, then there exists non negative numbers c_1 and c_2, such that they are not all zero and

$$c_1\left(\frac{u(x)}{v(y)}\right)^{\frac{1}{q}}v'(y)f^p(x)=c_2\left(\frac{v(y)}{u(x)}\right)^{\frac{1}{p}}u'(x)g^q(y)\ \text{ a.e. in } (a,b)\times(c,d).$$

It follows that

$$c_1\frac{u(x)}{u'(x)}f^p(x)=c_2\frac{v(y)}{v'(y)}g^q(y)=c_3\ \text{ a.e. in } (a,b)\times(c,d).$$

where c_3 is a constant.
Without loss of generality, suppose that $c_1\neq 0$. Then we have

$$\int_a^b f^p(x)dx=\frac{c_3}{c_1}\int_a^b\frac{u'(x)}{u(x)}dx=\frac{c_3}{c_1}\int_0^\infty\frac{1}{t}dt=\infty$$

which contradicts to the fact that $0 < \int_a^b f^p(x)dx < \infty$. Then by (2.6) and (2.7), we have

$$J < \left(\int_a^b \omega(u,v,q,x)(u(x))^{\frac{1}{q}} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_c^d \omega(v,u,p,y)(v(y))^{\frac{1}{p}} g^q(x)dx \right)^{\frac{1}{q}}$$

and in view of (2.8) and (2.9), it follows that (3.1) is valid.
For sufficiently small $\varepsilon > 0$, setting

$$f_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in (a, a_1)\ (a_1 = u^{-1}(1)) \\ (u(x))^{-\frac{1+\varepsilon}{p}} (u'(x))^{\frac{1}{p}} & \text{if } x \in [a_1, b) \end{cases}$$

$$g_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in (c, c_1)\ (c_1 = v^{-1}(1)) \\ (v(x))^{-\frac{1+\varepsilon}{q}} (v'(x))^{\frac{1}{q}} & \text{if } x \in [c_1, d) \end{cases}$$

we have

$$\left(\int_a^b f_\varepsilon^p(x)dx \right)^{\frac{1}{p}} \left(\int_c^d g_\varepsilon^q(x)dx \right)^{\frac{1}{q}} = \frac{1}{\varepsilon}. \tag{3.5}$$

If the constant factor $K(A,B)$ in (3.1) is not the best possible, then there exists a positive constant $C < K(A,B)$, such that (3.1) is still valid if we replace $K(A,B)$ by C. In particular by (2.10) and (3.5), we have

$$\begin{aligned} & K(A,B) + \circ(1) - \varepsilon \bigcirc (1) \\ & = \varepsilon I = \varepsilon \int_a^b \int_c^d \frac{(u'(x))^{\frac{1}{q}} (v'(y))^{\frac{1}{p}} f_\varepsilon(x) g_\varepsilon(y)}{A \min\{u(x), v(y)\} + B \max\{u(x), v(y)\}} dxdy \\ & < \varepsilon C \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}} \left(\int_c^d g^q(x)dx \right)^{\frac{1}{q}} \\ & = C \end{aligned}$$

and then $K(A,B) \leq C$ $(\varepsilon \to 0^+)$. This contradiction leads to the conclusion that the constant factor in (3.1) is the best possible. The theorem is proved. □

Remark 3.2. Taking $u(x) = v(x) = x$ in (3.2) and (3.3), we get the Hardy-Hilbert's inequality (1.1) and the Hardy-Hilbert type inequality (1.5) respectively. (1.9) recovered from (3.1) by taking $u(x) = v(x) = x$ and $p = q = 2$.

Taking suitable functions $u(x)$ and $v(x)$ in (3.2) and (3.3), we get many Hardy-Hilbert type inequalities.

Theorem 3.3. *Let* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, *and* $u \in \mathcal{F}(a,b)$. *If* $f \geq 0$ *satisfy* $0 < \int_a^b f^p(x)dx < \infty$, *then we obtain the equivalent inequality of* (3.1) *as* *follows:*

$$\begin{aligned}\int_c^d v'(y)\left[\int_a^b \frac{(u'(x))^{\frac{1}{q}}f(x)}{A\min\{u(x),v(y)\}+B\max\{u(x),v(y)\}}dx\right]^p dy \\ < (K(A,B))^p\int_a^b f^p(x)dx;\end{aligned} \tag{3.6}$$

where the constant factor $(K(A,B))^p$ *is the best possible. In particular,*

(i) *for* $A=B=1$*, we get an equivalent inequality of* (3.2) *as*

$$\int_c^d v'(y)\left[\int_a^b \frac{(u'(x))^{\frac{1}{q}}f(x)}{u(x)+v(y)}dx\right]^p dy < \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p\int_a^b f^p(x)dx; \tag{3.7}$$

(ii) *for* $A=0, B=1$*, we get an equivalent inequality of* (3.3) *as*

$$\int_c^d v'(y)\left[\int_a^b \frac{(u'(x))^{\frac{1}{q}}f(x)}{\max\{u(x),v(y)\}}dx\right]^p dy < (pq)^p\int_a^b f^p(x)dx; \tag{3.8}$$

where the constant factors $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p$ *and* $(pq)^p$ *are the best possible.*

Proof. Let $g(y)=(v'(y))^{\frac{1}{q}}\left[\int_a^b \frac{(u'(x))^{\frac{1}{q}}f(x)}{A\min\{u(x),v(y)\}+B\max\{u(x),v(y)\}}dx\right]^{p-1}$, then by (3.1), we have

$$\begin{aligned}0 &< \int_c^d g^q(y)dy \\ &= \int_c^d v'(y)\left[\int_a^b \frac{(u'(x))^{\frac{1}{q}}f(x)}{A\min\{u(x),v(y)\}+B\max\{u(x),v(y)\}}dx\right]^p dy \\ &= \int_a^b\int_c^d \frac{(u'(x))^{\frac{1}{q}}(v'(y))^{\frac{1}{p}}f(x)g(y)}{A\min\{u(x),v(y)\}+B\max\{u(x),v(y)\}}dxdy \\ &\le K(A,B)\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(\int_c^d g^q(y)dy\right)^{\frac{1}{q}},\end{aligned} \tag{3.9}$$

then

$$\begin{aligned}
0 &< \left(\int_c^d g^q(y)dy\right)^{\frac{1}{p}} \\
&= \left(\int_c^d v'(y)\left[\int_a^b \frac{(u'(x))^{\frac{1}{q}} f(x)}{A\min\{u(x),v(y)\}+B\max\{u(x),v(y)\}}dx\right]^p dy\right)^{\frac{1}{p}} \\
&\le K(A,B)\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \\
&< \infty.
\end{aligned} \tag{3.10}$$

It follows that (3.9) takes the form of strict inequality by using (3.1); so, does (3.10). Hence we can get (3.6).

On the other hand, if (3.6) holds, then by the Hölder's inequality, we have

$$\begin{aligned}
&\int_a^b\int_c^d \frac{(u'(x))^{\frac{1}{q}}(v'(y))^{\frac{1}{p}} f(x)g(y)}{A\min\{u(x),v(y)\}+B\max\{u(x),v(y)\}}dxdy \\
&= \int_c^d \left[(v'(y))^{\frac{1}{p}}\int_a^b \frac{(u'(x))^{\frac{1}{q}} f(x)}{A\min\{u(x),v(y)\}+B\max\{u(x),v(y)\}}dx\right] g(y)dy \\
&\le \left(\int_c^d v'(y)\left[\int_a^b \frac{(u'(x))^{\frac{1}{q}} f(x)}{A\min\{u(x),v(y)\}+B\max\{u(x),v(y)\}}dx\right]^p dy\right)^{\frac{1}{p}} \\
&\qquad\qquad \times \left(\int_c^d g^q(y)dy\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence by (3.6), (3.1) yields. Thus it follows that (3.1) and (3.6) are equivalent. Since the constant in (3.1) is the best possible, hence the constant in (3.6) is the best possible. The theorem is proved. □

4. THE DISCRETE VERSION OF THE INEQUALITY

Theorem 4.1. *Let* $p > 1, \frac{1}{p}+\frac{1}{q} = 1, m_0, n_0 \in \mathbb{N}, u \in \mathcal{F}(m_0-1,\infty)$ *and* $v \in \mathcal{F}(n_0-1,\infty)$ *such that* $u'(x)$ *and* $v'(x)$ *are decreasing in* (m_0-1,∞) *and* (n_0-1,∞) *respectively. If* $a_m, b_n \ge 0$ *satisfy* $0 < \sum_{m=m_0}^{\infty} a_m^p < \infty$ *and* $0 < \sum_{n=n_0}^{\infty} b_n^q < \infty$ *then*

$$\begin{aligned}
&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} \frac{(u'(m))^{\frac{1}{q}}(v'(n))^{\frac{1}{p}} a_m b_n}{A\min\{u(m),v(n)\}+B\max\{u(m),v(n)\}} \\
&\qquad < K(A,B)\left(\sum_{m=m_0}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty} b_n^q\right)^{\frac{1}{q}};
\end{aligned} \tag{4.1}$$

where the constant factor $K(A,B)$ is the best possible. In particular,

(i) *for $A=B=1$, we have*

$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{(u'(m))^{\frac{1}{q}}(v'(n))^{\frac{1}{p}}a_mb_n}{u(m)+v(n)}<\frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{m=m_0}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}b_n^q\right)^{\frac{1}{q}}; \tag{4.2}$$

(ii) *for $A=0, B=1$, we have*

$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{(u'(m))^{\frac{1}{q}}(v'(n))^{\frac{1}{p}}a_mb_n}{\max\{u(m),v(n)\}}<pq\left(\sum_{m=m_0}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}b_n^q\right)^{\frac{1}{q}}; \tag{4.3}$$

where the constant factors $\frac{\pi}{\sin\frac{\pi}{p}}$ and pq are the best possible.

Proof. Proceeding as in the theorem-3.1 and using Hölder's inequality, we get

$$\begin{aligned}&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{(u'(m))^{\frac{1}{q}}(v'(n))^{\frac{1}{p}}a_mb_n}{A\min\{u(m),v(n)\}+B\max\{u(m),v(n)\}}\\&\le\left\{\sum_{m=m_0}^{\infty}\left(\sum_{n=n_0}^{\infty}\frac{(v(n))^{-\frac{1}{q}}v'(n)}{A\min\{u(m),v(n)\}+B\max\{u(m),v(n)\}}\right)(u(m))^{\frac{1}{q}}a_m^p\right\}^{\frac{1}{p}}\\&\quad\times\left\{\sum_{n=n_0}^{\infty}\left(\sum_{m=m_0}^{\infty}\frac{(u(m))^{-\frac{1}{p}}u'(m)}{A\min\{u(m),v(n)\}+B\max\{u(m),v(n)\}}\right)(v(n))^{\frac{1}{p}}b_n^q\right\}^{\frac{1}{q}}.\end{aligned}$$

By (2.6) and (2.8), we get

$$\begin{aligned}&\sum_{n=n_0}^{\infty}\frac{(v(n))^{-\frac{1}{q}}v'(n)}{A\min\{u(m),v(n)\}+B\max\{u(m),v(n)\}}\\&<\int_{n_0-1}^{\infty}\frac{(v(y))^{-\frac{1}{q}}v'(y)}{A\min\{u(m),v(y)\}+B\max\{u(m),v(y)\}}dy\\&=\omega(u,v,q,m)=K(A,B)(u(m))^{-\frac{1}{q}}.\end{aligned}$$

Similarly, we get

$$\sum_{m=m_0}^{\infty}\frac{(u(m))^{-\frac{1}{p}}u'(m)}{A\min\{u(m),v(n)\}+B\max\{u(m),v(n)\}}<K(A,B)(v(n))^{-\frac{1}{p}}.$$

So, (4.1) is valid.
For sufficiently small $\varepsilon>0$, setting

$$\begin{aligned}\tilde{a}_m&=(u(m))^{-\frac{1+\varepsilon}{p}}(u'(m))^{\frac{1}{p}}\quad(m\ge m_0)\\\tilde{b}_n&=(v(n))^{-\frac{1+\varepsilon}{q}}(v'(n))^{\frac{1}{q}}\qquad(n\ge n_0).\end{aligned}$$

we have

$$\begin{aligned}\sum_{m=m_0}^{\infty} \tilde{a}_m^p &= \frac{u'(m_0)}{(u(m_0))^{1+\varepsilon}} + \sum_{m=m_0+1}^{\infty} \frac{u'(m)}{(u(m))^{1+\varepsilon}} \\ &\le \frac{u'(m_0)}{(u(m_0))^{1+\varepsilon}} + \int_{m_0}^{\infty} \frac{u'(x)}{(u(x))^{1+\varepsilon}} dx \qquad (4.4) \\ &= \frac{1}{\varepsilon}\left[\varepsilon \frac{u'(m_0)}{(u(m_0))^{1+\varepsilon}} + \frac{1}{(u(m_0))^{\varepsilon}}\right].\end{aligned}$$

Similarly,

$$\sum_{n=n_0}^{\infty} \tilde{b}_n^q \le \frac{1}{\varepsilon}\left[\varepsilon \frac{v'(n_0)}{(v(n_0))^{1+\varepsilon}} + \frac{1}{(v(n_0))^{\varepsilon}}\right]. \qquad (4.5)$$

Also proceeding as in the lemma-2.3, we have

$$\begin{aligned}&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} \frac{(u'(m))^{\frac{1}{q}}(v'(n))^{\frac{1}{p}}\tilde{a}_m\tilde{b}_n}{A\min\{u(m),v(n)\} + B\max\{u(m),v(n)\}} \\ &= \sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} \frac{(u(m))^{-\frac{1+\varepsilon}{p}}u'(m)(v(n))^{-\frac{1+\varepsilon}{q}}v'(n)}{A\min\{u(m),v(n)\} + B\max\{u(m),v(n)\}} \qquad (4.6) \\ &> \int_{m_0}^{\infty}\int_{n_0}^{\infty} \frac{(u(x))^{-\frac{1+\varepsilon}{p}}u'(x)(v(y))^{-\frac{1+\varepsilon}{q}}v'(y)}{A\min\{u(x),v(y)\} + B\max\{u(x),v(y)\}} dxdy \\ &> \frac{1}{\varepsilon}\frac{1}{(u(m_0))^{\varepsilon}}(K(A,B) + \circ(1)) - \bigcirc(1)\end{aligned}$$

If the constant factor $K(A,B)$ in (4.1) is not the best possible, then there exists a positive constant $C < K(A,B)$, such that (4.1) is still valid if we replace $K(A,B)$ by C. In particular by (4.4) , (4.5) and (4.6), we have

$$\begin{aligned}&\frac{1}{(u(m_0))^{\varepsilon}}(K(A,B) + \circ(1)) - \varepsilon \bigcirc (1) \\ &< \varepsilon \sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} \frac{(u'(m))^{\frac{1}{q}}(v'(n))^{\frac{1}{p}}\tilde{a}_m\tilde{b}_n}{A\min\{u(m),v(n)\} + B\max\{u(m),v(n)\}} \\ &< \varepsilon C \left(\sum_{m=m_0}^{\infty} \tilde{a}_m^p\right)^{\frac{1}{p}} \left(\sum_{n=n_0}^{\infty} \tilde{b}_n^q\right)^{\frac{1}{q}} \\ &= C\left(\varepsilon \frac{u'(m_0)}{(u(m_0))^{1+\varepsilon}} + \frac{1}{(u(m_0))^{\varepsilon}}\right)^{\frac{1}{p}} \left(\varepsilon \frac{v'(n_0)}{(v(n_0))^{1+\varepsilon}} + \frac{1}{(v(n_0))^{\varepsilon}}\right)^{\frac{1}{q}}\end{aligned}$$

and then $K(A,B) \le C$ ($\varepsilon \to 0^+$). This contradiction leads to the conclusion that the constant factor in (4.1) is the best possible. The theorem is proved. □

Remark 4.2. Taking $u(n) = v(n) = n$ in (4.2) and (4.3), we get the Hardy-Hilbert's inequality (1.3) and the Hardy-Hilbert type inequality (1.7) respectively.

Taking suitable functions $u(m)$ and $v(n)$ in (4.2) and (4.3), we get many Hardy-Hilbert type inequalities.

Theorem 4.3. *Let* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, m_0, n_0 \in \mathbb{N}, u \in \mathcal{F}(m_0 - 1, \infty)$ *and* $v \in \mathcal{F}(n_0 - 1, \infty)$ *such that* $u'(x)$ *and* $v'(x)$ *are decreasing in* $(m_0 - 1, \infty)$ *and* $(n_0 - 1, \infty)$ *respectively. If* $a_m \geq 0$ *satisfy* $0 < \sum_{m=m_0}^{\infty} a_m^p < \infty$ *then then we obtain the equivalent inequality of* (4.1) *as follows:*

$$\begin{aligned}\sum_{n=n_0}^{\infty} v'(n)\left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{\frac{1}{q}} a_m}{A\min\{u(m), v(n)\} + B\max\{u(m), v(n)\}}\right]^p \\ < (K(A,B))^p \sum_{m=m_0}^{\infty} a_m^p;\end{aligned} \tag{4.7}$$

where the constant factor $(K(A,B))^p$ *is the best possible. In particular,*

(i) *for* $A = B = 1$*, we get an equivalent inequality of* (4.2) *as*

$$\sum_{n=n_0}^{\infty} v'(n)\left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{\frac{1}{q}} a_m}{u(m) + v(n)}\right]^p < \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p \sum_{m=m_0}^{\infty} a_m^p; \tag{4.8}$$

(ii) *for* $A = 0, B = 1$*, we get an equivalent inequality of* (4.3) *as*

$$\sum_{n=n_0}^{\infty} v'(n)\left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{\frac{1}{q}} a_m}{\max\{u(m), v(n)\}}\right]^p < (pq)^p \sum_{m=m_0}^{\infty} a_m^p; \tag{4.9}$$

where the constant factors $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p$ *and* $(pq)^p$ *are the best possible.*

Proof. Taking $b_n = (v'(n))^{\frac{1}{q}}\left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{\frac{1}{q}} a_m}{A\min\{u(m),v(n)\}+B\max\{u(m),v(n)\}}\right]^{p-1}$ and proceeding as in the theorem-3.3, we prove the theorem. ☐

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