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On a Result of Mukherjee

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Abstract

For an arbitrary group G it is possible to introduce the notion of generalized supersolvably embedded normal subgroup of G. Related with this notion and with the notion of generalized hypercenter of G, there are some results in the finite case. Asaad and Mohamed (Comm.Algebra, 2001) showed that in a finite group G the generalized supersolvably embedded normal subgroup of G coincides with the generalized hypercenter of G. Here is shown how this result can be extended to the infinite case.

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1 Introduction and Statement of the Results

Two subgroups H and K of a group G are said to *permute* if $HK = KH$. It can be easily proved that H and K permute if and only if the set HK is a subgroup of G. A subgroup of G is said to be *quasinormal* (or *permutable*) in G if it permutes with every subgroup of G. Ore [7] defined the *quasicenter* $Q(G)$ of a group G to be the subgroup generated by all elements g of G such that $\langle g \rangle$ is *quasinormal* in G. Mukherjee [5] called *quasicentral* (*q.c.*) *elements* of G the elements of $Q(G)$. In [5] and [6] he introduced the hyperquasicenter of a group. The *hyperquasicenter* $Q_\infty(G)$ of a group G is the largest term of the chain

$$1 = Q_0(G) \leq Q_1(G) = Q(G) \leq Q_2(G) \leq \dots ,$$

where $Q_{\alpha+1}(G)/Q_\alpha(G) = Q(G/Q_\alpha(G))$ and

$$Q_\lambda(G) = \bigcup_{\alpha<\lambda} Q_\alpha(G),$$

α is an ordinal and λ is a limit ordinal.

We say, following Kegel [4], that a subgroup of G is *S-quasinormal* if it permutes with every Sylow subgroup of G. Agrawal [1] defined the *generalized center $genz(G)$* of G to be the subgroup of G generated by all elements g of G such that $\langle g\rangle$ is S-quasinormal in G. In analogy with Mukherjee [5], we call *Sylow-quasicentral* (S-*q.c.*) *elements* of G the elements of $genz(G)$. The *generalized hypercenter $genz_\infty(G)$* is the largest term of the chain

$$1 = genz_0(G) \leq genz_1(G) = genz(G) \leq genz_2(G) \leq \dots ,$$

where $genz_{\alpha+1}(G)/genz_\alpha(G) = genz(G/genz_\alpha(G))$ and

$$genz_\lambda(G) = \bigcup_{\alpha<\lambda} genz_\alpha(G),$$

α is an ordinal and λ is a limit ordinal.

We say that a normal subgroup H of a group G is *supersolvably embedded* in G if every chief factor of G which lies in H either has prime order or is infinite cyclic. Of course, if G is a finite group, then a normal subgroup H of G is supersolvably embedded in G if every chief factor of G has prime order. In this way we find the notion of Asaad and Mohamed [2, p.2239]. On another hand, embedding properties in Group Theory belongs to a long line of research as we can see in [9, Chapter 4]. In fact, given two normal subgroups H and K of a group G with $K \leq H$, we read in [9, p.217] that the factor group H/K is said to be *hypercyclically embedded* in G if to every normal subgroup N of G such that $K \leq N < H$, there exists a normal subgroup M of G such that $N < M \leq H$ and M/N is cyclic.

If G is a group and H is a normal subgroup of G, then the requirement for H to be hypercyclically embedded in G is stronger than requirement for H to be supersolvably embedded in G. If H and K are supersolvably (respectively hypercyclically) embedded in G, then HK is supersolvably (respectively hypercyclically) embedded in G [9, Lemma 5.2.1]. Therefore a group G has a unique maximal supersolvably (respectively hypercyclically) embedded subgroup $S(G)$ (respectively $\sigma(G)$). Of course, $\sigma(G)$ is contained in $S(G)$.

In [3] Beidleman and Heineken proved that $\sigma(G)$ coincides with $Q_\infty(G)$. Following Asaad and Mohamed [2], a normal subgroup H of a group G is *generalized supersolvably embedded* in G if there is a chain $1 = H_0 \leq H_1 \leq \dots$ of H such that H_α is S-quasinormal in G, $H_{\alpha+1}/H_\alpha$ is either cyclic of prime order or infinite cyclic, $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$, α is an ordinal and λ is a limit ordinal. The symbol $GSE(G)$ denotes the largest generalized supersolvably embedded subgroup of a group G.

If G is a finite group, Asaad and Mohamed [2, Theorem 3.5] proved that $GSE(G)$ coincides with $genz_\infty(G)$. In the proof of this result, they use Agrawal's Theorem [1] which characterizes a finite supersolvable group G as a group such that $G = genz_\infty(G)$. Our paper extends the result of Asaad and Mohamed [2, Theorem 3.5] to infinite groups, by showing that

Theorem A. *Let G be a group. Then $GSE(G) = genz_\infty(G)$.*

Theorem B. *Let G be group with nontrivial center and L be the subgroup of G generated by all the S-q.c. elements in G of finite order.*

(i) *L is periodic and hypercentral.*

(ii) *Let p be a prime and let L_p be the maximal p-subgroup of L. Then L_p is characteristic in G and generated by S-q.c. elements of p-power order.*

(iii) *If L_p has finite exponent p^n, where n is a positive integer, then we have $L_p \leq Z_{2n}(genz(G))$. Otherwise $L_p \leq Z_\omega(genz(G))$.*

Theorem C. *Let G be a group. Then there are two normal subgroups H and K of G such that*

(i) *H and K are generated by S-q.c. elements;*

(ii) *$genz(G) = HK$;*

(iii) *K' is abelian and $[K', H] = 1$;*

(iv) *$K' \leq Z_\omega(genz(G))$;*

(v) *$H \leq Z_\omega(genz(G))$.*

Our notation is standard and follows [8] and [9].

2 Preliminaries

Lemma 2.1. *Let G be a group. Then*

(i) $Z_\infty(G) \leq \sigma(G) = Q_\infty(G) \leq GSE(G) \leq genz_\infty(G)$;

(ii) $Z(G) \leq Q(G) \leq GSE(G)$;

(iii) $Z_\alpha(G) \leq Q_\alpha(G) \leq genz_\alpha(G)$ *for each ordinal* α.

(iv) $genz(G)$ *is locally nilpotent.*

Proof. The statements (i), (ii) and (iii) follow from the definitions and the fact that $\sigma(G) = Q_\infty(G)$ (see [3, Theorem 1]).

We will prove the statement (iv). If $genz(G) = Q(G)$, then the result is clear by [3, Lemma 3, (1)]. Now assume that $genz(G) \neq Q(G)$. Given $y \in genz(G)$, then $\langle y \rangle$ is ascendant in some Sylow subgroup P of G by [9, Theorem 6.2.10]. P is maximal in G so that $\langle y \rangle$ is ascendant in G. Thus each element of $genz(G)$ is properly contained in its normalizer in $genz(G)$ so that $genz(G)$ is an N-group (see [8, Chapter 6]) and $genz(G)$ is locally nilpotent by [8,Chapter 6,p.2].□

Lemma 2.2. *Let G be a group and H be a normal subgroup of G such that H is properly contained in $genz_\infty(G)$. Then*

(i) $genz(G/H)$ is nontrivial;

(ii) $genz_\infty(G/H) = genz_\infty(G)/H$.

Proof. (i). It is possible to consider an ordinal λ such that $genz_\lambda(G) \leq H$ and $genz_{\lambda+1}(G) \not\leq H$. Hence there is an element $y \in genz_{\lambda+1}(G) \setminus H$ such that $ygenz_\lambda(G)$ is an S-q.c. element of $G/genz_\lambda(G)$. Then yH is a nontrivial element of G/H as we want.

(ii). Since an epimorphic image of an S-quasinormal subgroup is again S-quasinormal (see [4]), $genz_1(G)H/H \leq genz_1(G/H)$. From this it is easy to see that $genz_2(G)H/H \leq genz_2(G/H)$. Let λ be an ordinal and assume that $genz_\alpha(G)H/H \leq genz_\alpha(G/H)$ for all ordinals $\alpha < \lambda$. If λ is not a limit ordinal, then $\lambda - 1$ exists and it is clear that $genz_\lambda(G)H/H \leq genz_\lambda(G/H)$. If λ is a limit ordinal, then it is straightforward to show that $genz_\lambda(G)H/H \leq genz_\lambda(G/H)$. Thus $genz_\infty(G)H/H \leq genz_\infty(G/H)$. Assume now that $genz_\infty(G)H/H$ is properly contained in $genz_\infty(G/H)$. By (i) $genz((G/H)/genz_\infty(G)/H)$ is nontrivial. Hence the same is true for $G/genz_\infty(G)$. This is a contradiction and (ii) follows.□

Proposition 2.3. *Let G be a group and let x and y be two elements in G such that x is an S-q.c. element of p-power order, with p a prime.*

(i) If y is of infinite order or of q-power order, where q is a prime distinct from p, then $y \in N_G(\langle x \rangle)$.

(ii) If y is of q-power order, q a prime distinct from p,and $(p-1, q) = 1$, then $[x, y] = 1$.

Proof. (i). Assume that y is of infinite order and consider $H = \langle x, y \rangle$. The intersection $\langle x \rangle \cap \langle y \rangle^p$ is identic, so if p is a prime, $S = \langle x, y^p \rangle$ has index p in H. Put $N = S_H$; then the index of N in H is finite and divides $p!$. Hence

$\langle x\rangle N/N \leq S/N$ and $\langle x\rangle N/N$ is a p'-subgroup of G/N. Since an epimorphic image of an S-quasinormal subgroup is again S-quasinormal, the subgroup $\langle x\rangle N/N$ is S-quasinormal in G/N. Thus if P/N is a Sylow subgroup of G/N, then $(\langle x\rangle N/N)^{P/N} = \langle x\rangle^P N/N$. It follows that $\langle x\rangle^P N \leq S$. Then $\langle x\rangle^P \leq S$ and $\langle x\rangle^P = \langle x\rangle\langle y^p\rangle$ for each prime p. This implies $y \in N_P(\langle x\rangle) \leq N_G(\langle x\rangle)$.

Now assume that y is of q-power order. By the famous Thompson-Feit's Theorem, $\langle x, y\rangle$ is solvable; since it is also finitely generated periodic, $\langle x, y\rangle$ is finite solvable. In particular $\langle x\rangle$ is a normal Sylow p-subgroup of $\langle x, y\rangle$.

(ii). This follows from (i) and from the fact that $\langle x, y\rangle$ is finite supersolvable.□

3 Proof of the results

Proof of Theorem A. By Lemma 2.1, $GSE(G) \leq genz_\infty(G)$. Assume that $GSE(G) \neq genz_\infty(G)$. We may suppose that $GSE(G) = 1$ without loss of generality. By Lemma 2.2 (i) $genz_\infty(G) \neq 1$. Then there exists a nontrivial S-q.c. element x of G. We have that $\langle x\rangle = X \neq 1$ is a subgroup of G such that the chain $1 = X_0 \leq X_1 = X$ has each term S-quasinormal in G and allows us to conclude that $X \leq GSE(G)$. Finally, $X = 1$ getting to contradiction. □

Proof of Theorem B. The fact that G has nontrivial center allows us to exclude the consideration of the periodic non-hypercentral groups of finite exponent.

(i). By Lemma 2.2, $genz(G)$ is locally nilpotent so it is clear that L and L_p are characteristic locally nilpotent subgroups of G. (ii). Let X denote the set of all S-q.c. elements of p-power order. Surely $\langle X\rangle \leq L_p$. Let $x \in L_p$ and let $x = x_1x_2\ldots x_n$, where each x_i is an S-q.c. element of G of finite order. By Lemma 2.2, $genz(G)$ is locally nilpotent and we can assume that $x_i \in L_p$ for each i. Hence $\langle X\rangle = L_p$ and we finish.

(iii). Let x be an S-q.c. element of G of p-power order. It is not hard to see that any subgroup of $\langle x\rangle$ is S-quasinormal in G. Thus there is an S-q.c. element x^+ of order p which is a power of x. The set of all these elements x^+ generates an abelian normal subgroup of G. We now show that x^+ normalizes each subgroup of $genz(G)$. For the subgroups that are contained in L and that are generated by elements of order prime to p, this is clear by the locally nilpotency of $genz(G)$. For p-subgroups A of L, it follows that A is normal in $\langle A, x^+\rangle = A\langle x^+\rangle$, since the index of A divides p. Consider now an S-q.c. element y of infinite order and put $Y = \langle y\rangle$. We know that the smallest normal subgroup X of $\langle x, y\rangle$ that contains x^+ is a periodic abelian group, so that $X \cap Y = 1$ and $\langle x^+, y\rangle = \langle x^+\rangle \times \langle y\rangle$. It follows that x^+ normalizes all the subgroups of $genz(G)$. By a result of Schenkman (see [9, Corollary 1.5.3]), $genz(G)$ must be contained in $Z_{2n}(genz(G))$. Hence both assertions follow. □

Proof of Theorem C. Let H be the subgroup of G generated by the S-q.c. elements of finite order and K the subgroup of G generated by the S-q.c. elements of infinite order. H and K are fixed by each automorphism of G, so they are normal in G.

By Lemma 2.2, $genz(G)$ is locally nilpotent, so every S-q.c. element x in G of infinite order normalizes every finite subgroup of $genz(G)$. For the same reason the commutator subgroup of the subgroup generated by two S-q.c. elements of infinite order is finite. This yields that commutators of two S-q.c. elements of infinite order centralize all elements of finite order in $genz(G)$. Then K' is abelian. The locally nilpotence of $genz(G)$ implies that elements of prime order and elements of infinite order in K centralize each other; in particular K' is contained in $Z(K)$. By induction it follows that $K' \leq Z_\omega(K)$, since K' is periodic. Finally $[K', H] = 1$ by Lemma 2.3. □

4 Examples

We note that the product of two S-q.c. elements is not an S-q.c. element as testified by [3, Section 2, First Example]. The same example shows how the product of two q.c. elements is not a q.c. element. Moreover relationships between the subgroups $Z(G)$, $Z_\infty(G)$, $Q(G)$, $Q_\infty(G)$, $S(G)$,$\sigma(G)$, $genz(G)$, $genz_\infty(G)$, $GSE(G)$ have been shown in the next examples.

Example 4.1. If A is the quaternion group

$$\langle a, b | a^4 = 1, b^2 = a^2, a^b = a^{-1}\rangle,$$

C is a cyclic group of order 9 with generator c and the action of C on A is given by $a^c = b, b^c = ab$, then the semidirect product G of A by C is a group of even order [2, Example 4]. Here we have

$$Z(G) = Q(G) = Q_\infty(G) = S(G) = \sigma(G) =$$

$$genz(G) = genz_\infty(G) = GSE(G) < G$$

and $|Z(G)| = 6$. Note that the cyclic subgroups of order 4 in A are permuted by C, none of them is permutable with any Sylow 3-subgroup of G.

Example 4.2. Let H be the alternating group on 4 elements, presented by

$$\langle a, b | a^3 = b^3 = (ab)^2 = 1 \rangle$$

and K the abelian free group of rank 3. We define the following action ϕ from H to $AutK \simeq GL(3, \mathbb{Z})$, via

$$a^\phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad b^\phi = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$(ab)^\phi = a^\phi b^\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The group $G = H \ltimes_\phi K$ has the presentation

$$\langle a, b, x, y, z : a^3 = b^3 = (ab)^2 = 1, [x, y] = [x, z] = [y, z] = 1,$$

$$x^a = y, y^a = z^{-1}, z^a = x, x^b = z, y^b = x, z^b = y^{-1} \rangle.$$

We have that

$$Q_\infty(G) = Q(G) = S(G) =$$

$$\sigma(G) = genz(G) = genz_\infty(G) = Z(G) = 1.$$

G is finitely generated solvable but it is not nilpotent.

Example 4.3. The infinite dihedral group $G = \langle a, x : a^x = a^{-1}, x^2 = 1 \rangle$ is the semidirect product of the infinite cyclic group $\langle a \rangle$ by the cyclic group $\langle x \rangle$ of order 2. By the presentation of G, it is clear that G has trivial center,

$$genz(G) = Q(G) = \langle a \rangle,$$

$$GSE(G) = genz_\infty(G) = Q_\infty(G) = S(G) = \sigma(G) = G.$$

Note that G possesses two conjugacy classes of Sylow 2-groups $\langle a^{2n+1}x \rangle$ and $\langle a^{2n}x \rangle$ (n is an integer), and $\langle x \rangle$ is not permutable with $\langle ax \rangle$.

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