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Fuzzy H-Ideals of BCI-Algebras with Interval Valued Membership Functions

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Abstract. The purpose of this paper is to define the notion of an interval-valued fuzzy H-ideal(briefly, an $i-v$ fuzzy H-ideal) of a BCI-algebra. Necessary and sufficient conditions for an $i-v$ fuzzy ideal to be an $i-v$ fuzzy H-ideal are stated. A way to make a new $i-v$ fuzzy H-ideal from old one is given.

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1. Preliminaries

The notion of BCK-algebras was proposed by Iami and Is'eki in 1966. In the same year, Is'eki [4] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras. Fuzzy sets were initiated by Zadeh [8]. In [7], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as an $i-v$ fuzzy set. In [7], Zadeh also constructed a method of approximate inference using his $i-v$ fuzzy sets. In [2], Biswas defined interval-valued fuzzy subgroups (i.e., $i-v$ fuzzy subgroups) of Rosenfeld's

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nature, and investigated some elementary properties. In this paper, using the notion of interval-valued fuzzy set by Zadeh, we introduce the concept of an interval-valued fuzzy BCI-subalgebra (briefly, $i-v$ fuzzy BCI-subalgebra) of a BCI-algebra, and study some of their properties. Using an $i-v$ level set of an $i-v$ fuzzy set, we state a characterization of an $i-v$ fuzzy H-ideal of BCI-algebras. We prove that every $i-v$ fuzzy H-ideal of a BCI-algebra X can be realized as an $i-v$ level H-ideal of an $i-v$ fuzzy H-ideal of X. In connection with the notion of homomorphism, we study how the images and inverse images of $i-v$ fuzzy H-ideal become $i-v$ fuzzy H-ideal.

By a BCI-algebra we mean an algebra $(X;*,0)$ of type $(2,0)$ satisfying the following axioms:

(1) $((x*y)*(x*z))*(z*y)=0$,
(2) $(x*(x*y))*y)=0$,
(3) $x*x=0$,
(4) $x*y=0$ and $y*x=0$ imply $x=y$.

for all $x,y,z\in X$. We can define a partial ordering $''\leq''$ on X by $x\leq y$ if and only if $x*y=0$.

The following statements are true in any BCI-algebra X:

(1.1) $(x*y)*z=(x*z)*y$,
(1.2) $x*0=x$,
(1.3) $(x*z)*(y*z)\leq x*y$,
(1.4) $x\leq y$ implies $x*z\leq y*z$ and $z*y\leq z*x$,
(1.5) $0*(x*y)=(0*x)*(0*y)$,
(1.6) $x*(x*(x*y))=x*y$.

Definition 1. A non empty subset I of X is called an ideal of X if it satisfies:
(I_1) $0\in I$,
(I_2) $x*y\in I$ and $y\in I$ imply $x\in I$.

Definition 2. A nonempty subset I of X is called an H-ideal of X if it satisfies condition (I_1) and
(I_3) $x*(y*z)\in I$ and $y\in I$ imply $x*z\in I$.

Putting $z=0$ in (I_3) then we can see that every H-ideal is an ideal.

Definition 3. A fuzzy set μ in a BCI-algebra X is called a fuzzy H-ideal of X if
(FI_1) $\mu(0)\geq\mu(x)$,
(FI_2) $\mu(x*z)\geq\min\{\mu(x*(y*z)),\mu(y)\}$.

An interval-valued fuzzy set (briefly, $i-v$ fuzzy set) A defined on X is given by

$$A = \{(x, [\mu_A^L(x), \mu_A^U(x)])\}, \quad \forall x \in X (briefly, denoted \ \ by \ A = [\mu_A^L, \mu_A^U]),$$

where μ_A^L and μ_A^U are two fuzzy sets in X such that $\mu_A^L \leq \mu_A^U$ for all $x \in X$. Let $\overline{\mu}_A(x) = [\mu_A^L, \mu_A^U], \forall x \in X$ and let $D[0,1]$ denotes the family of all closed subintervals of $[0,1]$. If $\mu_A^L(x) = \mu_A^U(x) = c$, where $0 \leq c \leq 1$, say then we have $\overline{\mu}_A(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to $D[0,1]$. Thus $\overline{\mu}_A(x) \in D[0,1], \forall x \in X$, and therefore the $i-v$ fuzzy set A is given by

$$A = \{(x, \overline{\mu}_A(x))\}, \forall x \in X, where \ \overline{\mu}_A(x) : X \to D[0,1].$$

Now let us define what is known as refined minimum (briefly, rmin) of two elements in $D[0,1]$. We also define the symbols " $\geq$ "," $\leq$ ", and " $=$ " in case of two elements in $D[0,1]$. Consider two elements $D_1 := [a_1, b_1]$ and $D_2 := [a_2, b_2] \in D[0,1]$. Then

$$rmin(D_1, D_2) = [min\{a_1, a_2\}, min\{b_1, b_2\}]$$

$$D_1 \geq D_2 \Leftrightarrow a_1 \geq a_2, b_1 \geq b_2;$$

and similarly we may have $D_1 \leq D_2$ and $D_1 = D_2$.

2. INTERVAL-VALUE FUZZY H-IDEALS OF BCI-ALGEBRAS

Definition 4. An Interval valued fuzzy set A in BCI-algebra X is called an interval-valued fuzzy H-ideal of X if it satisfies

(FI_1) $\overline{\mu}_A(0) \geq \overline{\mu}_A(x)$

(FI_2) $\overline{\mu}_A(x * z) \geq rmin\{\overline{\mu}_A(x * (y * z)), \overline{\mu}_A(y)\}$.

Example 1. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

*	0	a	b	c	d	e
0	0	0	0	c	c	c
a	a	0	a	d	c	d
b	b	b	0	e	e	c
c	c	c	c	0	0	0
d	d	c	d	a	0	a
e	e	e	c	b	b	0

let an $i-v$ fuzzy set A defined on X be given by

$$\overline{\mu}_A(x) = \begin{cases} [0.4, 0.9] & ; \quad x \in \{0, a\} \\ [0.1, 0.3] & ; \quad othewise \end{cases}$$

It is easy to check that A is an $i-v$ fuzzy H-ideal of BCI-algebra X.

Theorem 1. *Let A be an $i-v$ fuzzy H-ideal of X. If there is a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}\overline{\mu}_A(x_n)=[1,1]$ then $\overline{\mu}_A(0)=[1,1]$.*

Proof. Since $\overline{\mu}_A(0)\geq\overline{\mu}_A(x)$ for all $x\in X$, we have $\overline{\mu}_A(0)\geq\overline{\mu}_A(x_n)$ for every positive integer n. Note that

$$[1,1]\geq\overline{\mu}_A(0)\geq\lim_{n\to\infty}\overline{\mu}_A(x_n)=[1,1].$$

Hence $\overline{\mu}_A(0)=[1,1]$.

Theorem 2. *An $i-v$ fuzzy set $A=[\mu_A^L,\mu_A^U]$ in X is an $i-v$ fuzzy H-ideal of X if and only if μ_A^L and μ_A^U are fuzzy H-ideal of X.*

Proof. Since $\mu_A^L(0)\geq\mu_A^L(x)$ and $\mu_A^U(0)\geq\mu_A^U(x)$, therefore $\overline{\mu}_A(0)\geq\overline{\mu}_A(x)$. Suppose that μ_A^L and μ_A^U are fuzzy H-ideal of X. Let $x,y,z\in X$, then

$$\begin{aligned}\overline{\mu}_A(x*z)&=[\mu_A^L(x*z),\mu_A^U(x*z)]\\&\geq[\min\{\mu_A^L(x*(y*z)),\mu_A^L(y)\},\min\{\mu_A^U(x*(y*z)),\mu_A^U(y)\}]\\&=rmin\{[\mu_A^L(x*(y*z)),\mu_A^U(x*(y*z))],[\mu_A^L(y),\mu_A^U(y)]\}\\&=rmin\{\overline{\mu}_A(x*(y*z)),\overline{\mu}_A(y)\}.\end{aligned}$$

Hence A is an $i-v$ fuzzy H-ideal of X.
Conversely, assume that A is an $i-v$ fuzzy H-ideal of X. For any $x,y\in X$, we have

$$\begin{aligned}[\mu_A^L(x*z),\mu_A^U(x*z)]&=\overline{\mu}_A(x*z)\\&\geq rmin\{\overline{\mu}_A(x*(y*z)),\overline{\mu}_A(y)\}\\&=rmin\{[\mu_A^L(x*(y*z)),\mu_A^U(x*(y*z))],[\mu_A^L(y),\mu_A^U]\}\\&=[min\{\mu_A^L(x*(y*z)),\mu_A^L(y)\},min\{\mu_A^U(x*(y*z)),\mu_A^U(y)\}].\end{aligned}$$

It follows that $\mu_A^L(x*z)\geq min\{\mu_A^L(x*(y*z)),\mu_A^L(y)\}$ and $\mu_A^U(x*z)\geq min\{\mu_A^U(x*(y*z)),\mu_A^U(y)\}$. Hence μ_A^L and μ_A^L are fuzzy H-ideal of X.

Lemma 1. *An $i-v$ fuzzy set $A=[\mu_A^L,\mu_A^U]$ in X is an $i-v$ fuzzy ideal of X if and only if μ_A^L and μ_A^U are fuzzy ideals of X.*

Proof. Since $\mu_A^L(0)\geq\mu_A^L(x)$ and $\mu_A^U(0)\geq\mu_A^U(x)$, therefore $\overline{\mu}_A(0)\geq\overline{\mu}_A(x)$. Suppose that μ_A^L and μ_A^U are fuzzy ideals of X. Let $x,y\in X$, then

$$\begin{aligned}\overline{\mu}_A(x)&=[\mu_A^L(x),\mu_A^U(x)]\\&\geq[\min\{\mu_A^L(x*y),\mu_A^L(y)\},\min\{\mu_A^U(x*y),\mu_A^U(y)\}]\\&=rmin\{[\mu_A^L(x*y),\mu_A^U(x*y)],[\mu_A^L(y),\mu_A^U(y)]\}\\&=rmin\{\overline{\mu}_A(x*y),\overline{\mu}_A(y)\}.\end{aligned}$$

Hence A is an $i-v$ fuzzy ideal of X.
Conversely, assume that A is an $i-v$ fuzzy ideal of X. For any $x,y\in X$, we have

$$\begin{aligned}[\mu_A^L(x), \mu_A^U(x)] = \overline{\mu}_A(x) &\\ &\geq rmin\{\overline{\mu}_A(x*y), \overline{\mu}_A(y)\} \\ &= rmin\{[\mu_A^L(x*y), \mu_A^U(x*y)], [\mu_A^L(y), \mu_A^U(y)]\} \\ &= [min\{\mu_A^L(x*y), \mu_A^L(y)\}, min\{\mu_A^U(x*y), \mu_A^U(y)\}].\end{aligned}$$

It follows that $\mu_A^L(x) \geq min\{\mu_A^L(x*y), \mu_A^L(y)\}$ and $\mu_A^U(x) \geq min\{\mu_A^U(x*y), \mu_A^U(y)\}$. Hence μ_A^L and μ_A^L are fuzzy ideals of X.

Theorem 3. *Every $i-v$ fuzzy H-ideal of a BCI-algebra X is an $i-v$ fuzzy ideal.*

Proof. Let $A = [\mu_A^L, \mu_A^U]$ be an $i-v$ fuzzy H-ideals of X, where μ_A^L and μ_A^U are two fuzzy H-ideals of BCI-algebra X. Thus μ_A^L and μ_A^U, by Proposition 1 of [5], are fuzzy ideals of X. Hence by lemma 1, A is an $i-v$ fuzzy ideal of X.

Theorem 4. *Let $A = [\mu_A^L, \mu_A^U]$ be a $i-v$ fuzzy set in a BCI-algebra X. Then the following statements are equivalent:*

(i) A is an $i-v$ fuzzy H-ideal of X,

(ii) for all $x, y \in X$; $\overline{\mu}_A(x*y) \geq \overline{\mu}_A(x*(0*y))$,

(ii) for all $x, y, z \in X$; $\overline{\mu}_A((x*y)*z) \geq \overline{\mu}_A(x*(y*z))$.

Proof. $(i) \Rightarrow (ii)$ Let $\overline{\mu}_A$ be an $i-v$ fuzzy fuzzy H-ideal of X, then

$$\overline{\mu}_A(x*y) \geq rmin\{\overline{\mu}_A(x*(0*y)), \overline{\mu}_A(0)\} = \overline{\mu}_A(x*(0*y)).$$

Therefore $\overline{\mu}_A(x*y) \geq \overline{\mu}_A(x*(0*y))$ for all $x, y \in X$.

$(ii) \Rightarrow (iii)$ For all $x, y, z \in X$, we have

$$\begin{aligned}((x*y)*(0*z))*(x*(y*z)) &= ((x*y)*(x*(y*z)))*(0*z) \\ &\leq ((y*z)*y)*(0*z) \\ &= (0*z)*(0*z) = 0,\end{aligned}$$

hence $(x*y)*(0*z) \leq x*(y*z)$. Therefore

$$\begin{aligned}\overline{\mu}_A((x*y)*(0*z)) &= [\mu_A^L((x*y)*(0*z)), \mu_A^U((x*y)*(0*z))] \\ &= [\mu_A^L(x*(y*z)), \mu_A^U(x*(y*z))] \\ &= \overline{\mu}_A(x*(y*z)).\end{aligned}$$

Thus by (ii), we have $\overline{\mu}_A((x*y)*z) \geq \overline{\mu}_A(x*(y*z))$.

$(ii) \Rightarrow (iii)$ For all $x, y, z \in X$, we have

$$\begin{aligned}\overline{\mu}_A(x*z) &\geq rmin\{\overline{\mu}_A((x*z)*y), \overline{\mu}_A(y)\} \\ &= rmin\{\overline{\mu}_A((x*y)*z), \overline{\mu}_A(y)\} \\ &\geq rmin\{\overline{\mu}_A(x*(y*z)), \overline{\mu}_A(y)\}.\end{aligned}$$

Therefore $\overline{\mu}_A(x*z) \geq rmin\{\overline{\mu}_A(x*(y*z)), \overline{\mu}_A(y)\}$. Hence A is an $i-v$ fuzzy H-ideal of X.

Theorem 5. *Let A be an $i-v$ fuzzy set in a BCI-algebra X. Then A is an $i-v$ fuzzy H-ideal of X if and only if the nonempty set*

$$\overline{U}(A; [\delta_1, \delta_2]) := \{x \in X | \overline{\mu}_A(x) \geq [\delta_1, \delta_2]\}$$

is a H-ideal of X for every $[\delta_1, \delta_2] \in D[0, 1]$. We then call $\overline{U}(A; [\delta_1, \delta_2])$ the $i-v$ level H-ideal of X.

Proof. Assume that A is an $i-v$ fuzzy H-ideal of X. Since for all $x \in \overline{U}(A;[\delta_1,\delta_2])$ we have $\overline{\mu}_A(0) \geq \overline{\mu}_A(x) \geq [\delta_1,\delta_2]$. Therefore $0 \in \overline{U}(A;[\delta_1,\delta_2])$. Now, let $x,y,z \in X$ such that, $x*(y*z), y \in \overline{U}(A;[\delta_1,\delta_2])$. Then

$$\begin{aligned}\overline{\mu}_A(x*z) &\geq rmin\{\overline{\mu}_A(x*(y*z)), \overline{\mu}_A(y)\}\\ &\geq rmin\{[\delta_1,\delta_2],[\delta_1,\delta_2]\}\\ &= [\delta_1,\delta_2],\end{aligned}$$

and so $x*z \in \overline{U}(A;[\delta_1,\delta_2])$. Thus $\overline{U}(A;[\delta_1,\delta_2])$ is a H-ideal of BCI-algebra of X.

Conversely, assume that $\overline{U}(A;[\delta_1,\delta_2])(\neq \emptyset)$ is a H-ideal of X for every $[\delta_1,\delta_2] \in D[0,1]$. Suppose that there exist $x_0,y_0,z_0 \in X$ such that $\overline{\mu}_A(x_0*z_0) < rmin\{\overline{\mu}_A(x_0*(y_0*z_0)), \overline{\mu}_A(y_0)\}$.

Let $\overline{\mu}_A(x_0*(y_0*z_0)) = [\beta_1,\beta_2]$,$\overline{\mu}_A(y_0) = [\beta_3,\beta_4]$, and $\overline{\mu}_A(x_0*z_0) = [\delta_1,\delta_2]$, then

$$\begin{aligned}[\delta_1,\delta_2] &< rmin\{[\beta_1,\beta_2],[\beta_3,\beta_4]\\ &= [\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\}]\end{aligned}$$

Hence $\delta_1 < \min\{\beta_1,\beta_3\}$ and $\delta_2 < \min\{\beta_2,\beta_4\}$. Taking

$$[\lambda_1,\lambda_2] = \frac{1}{2}(\overline{\mu}_A(x_0*z_0) + rmin\{\overline{\mu}_A(x_0*(y_0*z_0)), \overline{\mu}_A(y_0)\}),$$

we obtain

$$\begin{aligned}[\lambda_1,\lambda_2] &= \tfrac{1}{2}([\delta_1,\delta_2] + [\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\}])\\ &= [\tfrac{1}{2}(\delta_1+\min\{\beta_1,\beta_3\}), \tfrac{1}{2}(\delta_2+\min\{\beta_2,\beta_4\})].\end{aligned}$$

It follows that

$$\min\{\beta_1,\beta_3\} > \beta_1 = \frac{1}{2}(\delta_1+\min\{\beta_1,\beta_3\}) > \delta_1,$$

$$\min\{\beta_2,\beta_4\} > \beta_2 = \frac{1}{2}(\delta_2+\min\{\beta_2,\beta_4\}) > \delta_2,$$

so that $[\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\}] > [\beta_1,\beta_2] > [\delta_1,\delta_2] = \overline{\mu}_A(x_0*z_0)$.

Therefore, $x_0*z_0 \notin \overline{U}(A,[\beta_1,\beta_2])$.

On the other hand,

$$\overline{\mu}_A(x_0*(y_0*z_0)) = [\beta_1,\beta_2] \geq [\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\}] > [\beta_1,\beta_2]$$

and

$$\overline{\mu}_A(y_0) = [\beta_3,\beta_4] \geq [\min\{\beta_1,\beta_3\},\min\{\beta_2,\beta_4\}] > [\beta_1,\beta_2]$$

and so $x_0*(y_0*z_0), y_0 \in \overline{U}(A,[\beta_1,\beta_2])$.

But this contradicts the fact that $\overline{U}(A,[\beta_1,\beta_2])$ is a H-ideal of X. Hence $\overline{\mu}_A(x*z) \geq rmin\{\overline{\mu}_A(x*(y*z)), \overline{\mu}_A(y)\}$, for all $x,y,z \in X$.

Theorem 6. *Every H-ideal of a BCI-algebra X can be realized as an $i-v$ level H-ideal of an $i-v$ fuzzy H-ideal of X.*

Proof. Let I be a H-ideal and A be an $i-v$ fuzzy set on X defined by

$$\overline{\mu}_A(x) = \begin{cases} [\alpha_1, \alpha_2] & ; \quad x \in I \\ [0, 0] & ; \quad otherwise \end{cases}$$

where $\alpha_1, \alpha_2 \in (0, 1]$, with $\alpha_1 < \alpha_2$. We show that A is an $i-v$ fuzzy H-ideal of X. Let $x*(y*z), y \in I$, then $x*z \in I$ and so

$$\overline{\mu}_A(x*z) = [\alpha_1, \alpha_2] = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = rmin\{\overline{\mu}_A(x*(y*z)), \overline{\mu}_A(y)\}.$$

If at least one of $x*(y*z)$ and y is not in A, then at least one of $\overline{\mu}_A(x*(y*z))$ and $\overline{\mu}_A(y)$ is 0. Therefore

$$\overline{\mu}_A(x*z) \geq [0, 0] = rmin\{[0, 0], [0, 0]\} = rmin\{\overline{\mu}_A(x*(y*z)), \overline{\mu}_A(y)\}$$

This means that $\overline{\mu}_A$ satisfies (FI_2). On the other hands, since $0 \in I, \overline{\mu}_A(0) = [\alpha_1, \alpha_2] \geq \overline{\mu}_A(x)$, for all $x \in X$ and so $\overline{\mu}_A$ satisfies (FI_1). Thus, $\overline{\mu}_A$ is an $i-v$ fuzzy H-ideal of X. It is clear that $\overline{U}(A; [\alpha_1, \alpha_2]) = I$. This completes the proof.

Theorem 7. *Let I be a subset of a BCI-algebra X, such that $0 \in I$ and let A be an $i-v$ fuzzy set on X which is given in the proof of Theorem 5. If A is an $i-v$ fuzzy H-ideal of X, then I is a H-ideal of X.*

Proof. Assume that A is an $i-v$ fuzzy H-ideal of X. Let $x*(y*z), y \in I$, then $\overline{\mu}_A(x*(y*z)) = [\alpha_1, \alpha_2] = \overline{\mu}_A(y)$, and so,

$$\overline{\mu}_A(x*z) \geq rmin\{\overline{\mu}_A(x*(y*z)), \overline{\mu}_A(y)\} = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2].$$

This implies that $x*z \in I$.

Theorem 8. *If A is an $i-v$ fuzzy H-ideal of a BCI-algebra X, then the set $X_{\overline{\mu}_A} := \{x \in X | \overline{\mu}_A(x) = \overline{\mu}(0)\}$ is a H-ideal of X*

Proof. Let $x*(y*z), y \in X_{\overline{\mu}_A}$. Then $\overline{\mu}_A(y) = \overline{\mu}_A(0) = \overline{\mu}_A(x*(y*z))$, and so

$$\overline{\mu}_A(x*z) \geq rmin\{\overline{\mu}_A(x*(y*z)), \overline{\mu}_A(y)\} = rmin\{\overline{\mu}_A(0), \overline{\mu}_A(0)\} = \overline{\mu}_A(0).$$

Therefore $\overline{\mu}_A(x*z) = \overline{\mu}_A(0)$, that is $x*z \in X_{\overline{\mu}_A}$. Hence $X_{\overline{\mu}_A}$ is a H-ideal of X.

Theorem 9. *For an $i-v$ fuzzy H-ideal A of BCI-algebra X, the $i-v$ fuzzy set A^* in X defined by $\overline{\mu}_{A^*}(x) = \overline{\mu}_A(0*x)$, for all $x \in X$ is an $i-v$ fuzzy H-ideal of X.*

Proof. For all $x, y, z \in X$, we have

$$\begin{aligned} \overline{\mu}_{A^*}(x*z) &= \overline{\mu}_A(0*(x*z)) \\ &= \overline{\mu}_A((0*x)*(0*z)) \\ &\geq rmin\{\overline{\mu}_A((0*x)*((0*y)*(0*z))), \overline{\mu}_A(0*y)\} \\ &= rmin\{\overline{\mu}_A(0*(x*(y*z))), \overline{\mu}_A(0*y)\} \\ &= rmin\{\overline{\mu}_{A^*}(x*(y*z)), \overline{\mu}_{A^*}(y)\}. \end{aligned}$$

Therefore A^* is an $i-v$ fuzzy H-ideal of X.

Theorem 10. *Let A be an $i-v$ fuzzy ideal of BCI-algebra X. If $\overline{\mu}_A(x*y) \geq \overline{\mu}_A(x)$ for all $x, y \in X$, then A is an $i-v$ fuzzy H-ideal of X.*

Proof. Since A is an $i-v$ fuzzy ideal of X, by hypothesis we have

$rmin\{\overline{\mu}_A(x*(y*z)), \overline{\mu}_A(y)\} \leq rmin\{\overline{\mu}_A((x*z)*(y*z)), \overline{\mu}_A(y*z)\} \leq \overline{\mu}_A(x*z).$

For all $x, y, z \in X$. Hence A is an $i-v$ fuzzy H-ideal of X.

Definition 5. ([3]) An $i-v$ fuzzy set A in X is called an interval-valued fuzzy BCI-subalgebra (briefly, $i-v$ fuzzy BCI-subalgebra) of X if

$$\overline{\mu}_A(x*y) \geq rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} \ \ \forall x, y \in X.$$

Theorem 11. *Every $i-v$ fuzzy H-ideal of BCI-algebra X is an $i-v$ fuzzy subalgebra of X.*

Proof. Let $A = [\mu_A^L, \mu_A^U]$ be an $i-v$ fuzzy H-ideal of X, where μ_A^L and μ_A^U are two fuzzy H-ideal of BCI-algebra X. Thus μ_A^L and μ_A^U are fuzzy subalgebra of X. Hence by Theorem 3.7 of [3], A is an $i-v$ fuzzy subalgebra of X.

3. Cartesian product of $i-v$ fuzzy H-ideals

Definition 6. (see also [1]). A fuzzy relation A on any set X is a fuzzy subset A with a membership function $\Omega_A : X \times X \to [0, 1]$.

Lemma 2. *(see also [1]) Let $\overline{\mu}_A$ and $\overline{\mu}_B$ be membership function of each $x \in X$ to the $i-v$ subsets A and B, respectively. Then $\mu_A \times \mu_B$ is membership function of each element $(x, y) \in X \times X$ to the set $A \times B$ and defined by*
$(\overline{\mu}_A \times \overline{\mu}_B)(x, y) = rmin\{\overline{\mu}_A(x), \overline{\mu}_B(y)\}$

Definition 7. Let $A = [\mu_A^L, \gamma_A^U]$ and $B = [\mu_B^L, \gamma_B^U]$ be two $i-v$ fuzzy subsets in a set X. The cartesian product of A and B is defined by

$A \times B = \{((x, y), \overline{\mu}_A \times \overline{\mu}_B); \forall (x, y) \in X \times X$, where $\overline{\mu}_A \times \overline{\mu}_B : X \times X \to D[0, 1]$.

Theorem 12. *Let $A = [\mu_A^L, \mu_A^U]$ and $B = [\mu_B^L, \mu_B^U]$ be two $i-v$ fuzzy subsets in a set X, then $A \times B$ is an $i-v$ fuzzy H-ideal of $X \times X$.*

Proof. Let $(x, y) \in X \times X$, then by definition

$$\begin{aligned}(\overline{\mu}_A \times \overline{\mu}_B)(0, 0) &= rmin\{\overline{\mu}_A(0), \overline{\mu}_B(0)\} \\ &= rmin\{[\mu_A^L(0), \mu_A^U(0)], [\mu_B^L(0), \mu_B^U(0)]\}\end{aligned}$$

$$\begin{aligned}
&= [\min\{\mu_A^L(0), \mu_B^L(0)\}, \min\{\mu_A^U(0), \mu_B^U(0)\}] \\
&\geq [\min\{\mu_A^L(x), \mu_B^L(y)\}, \min\{\mu_A^U(x), \mu_B^U(y)\}] \\
&= rmin\{[\mu_A^L(x), \mu_A^U(x)], [\mu_B^L(y), \mu_B^U(y)]\} \\
&= rmin\{\overline{\mu}_A(x), \overline{\mu}_B(y)\} \\
&= (\overline{\mu}_A \times \overline{\mu}_B)(x, y).
\end{aligned}$$

Therefore (FI_2), holds.
Now, for all $x, y, z \in X$, we have

$$\begin{aligned}
(\overline{\mu}_A \times \overline{\mu}_B)((x, \acute{x}) * (z, \acute{z})) &= (\overline{\mu}_A \times \overline{\mu}_B)(x * z, \acute{x} * \acute{z}) \\
&= rmin\{\mu_A(x * z), \mu_B(\acute{x} * \acute{z})\} \\
&\geq rmin\{rmin\{\overline{\mu}_A(x * (y * z)), \overline{\mu}_A(y)\}, rmin\{\overline{\mu}_B(\acute{x} * (\acute{y} * \\
&\quad \acute{z})), \overline{\mu}_B(\acute{y})\}\} \\
&= rmin\{[min\{\mu_A^L(x * (y * z)), \mu_A^L(y)\}, min\{\mu_A^U(x * (y * \\
&\quad z)), \mu_A^U(y)\}], [min\{\mu_B^L(\acute{x} * (\acute{y} * \acute{z})), \mu_B^L(\acute{y})\}, min\{\mu_B^U(\acute{x} * \\
&\quad (\acute{y} * \acute{z})), \mu_B^U(\acute{y})\}]\} \\
&= [min\{min\{\mu_A^L(x*(y*z)), \mu_B^L(\acute{x}*(\acute{y}*\acute{z})\}, min\{\mu_A^L(y), \mu_B^L(\acute{y})\}\}\}, \\
&\quad min\{min\{\mu_A^U(x*(y*z)), \mu_B^U(\acute{x}*(\acute{y}*\acute{z})\}, min\{\mu_A^U(y), \mu_B^U(\acute{y})\}\}\}] \\
&= rmin\{(\overline{\mu}_A \times \overline{\mu}_B)(x * (y * z), \acute{x} * (\acute{y} * \acute{z})), (\overline{\mu}_A \times \overline{\mu}_B)(y, \acute{y})\}.
\end{aligned}$$

Hence $A \times B$ is an $i - v$ fuzzy H-ideal of $X \times X$.

Theorem 13. *Let $A = [\mu_A^L, \gamma_A^U]$ and $B = [\mu_B^L, \gamma_B^U]$ be two $i - v$ of a set X. If $A \times B$ is $i - v$ fuzzy H-ideal of $X \times X$, then*
(i) $\overline{\mu}_A(0) \geq \overline{\mu}_A(x)$ or $\overline{\mu}_B(0) \geq \overline{\mu}_B(x)$,
(ii) $\overline{\mu}_B(0) \geq \overline{\mu}_A$ or $\overline{\mu}_B(0) \geq \overline{\mu}_B(x)$,
(iii) $\overline{\mu}_A(0) \geq \overline{\mu}_A(x)$ or $\overline{\mu}_A(0) \geq \overline{\mu}_B(x)$,
(iv) A or B is an $i - v$ fuzzy H-ideal of X.

Proof.(i) Suppose that $\overline{\mu}_A(0) < \overline{\mu}_A(x)$ and $\overline{\mu}_B(0) < \overline{\mu}_B(x)$, then

$$\begin{aligned}
(\overline{\mu}_A \times \overline{\mu}_B)(x, y) &= rmin\{\overline{\mu}_A(x), \overline{\mu}_B(y)\} \\
&= [min\{\mu_A^L(x), \mu_B^L(y)\}, min\{\mu_A^U(x), \mu_B^U(y)\}] \\
&> [min\{\mu_A^L(0), \mu_B^L(0)\}, min\{\mu_A^U(0), \mu_B^U(0)\}] \\
&= rmin\{\overline{\mu}_A(0), \overline{\mu}_B(0)\} = (\overline{\mu}_A \times \overline{\mu}_B)(0, 0).
\end{aligned}$$

This is a contraction.
(ii) Suppose that $\overline{\mu}_B(0) < \overline{\mu}_A(x)$ and $\overline{\mu}_B(0) < \overline{\mu}_B(x)$, then

$$\begin{aligned}
(\overline{\mu}_A \times \overline{\mu}_B)(0, 0) &= rmin\{\overline{\mu}_A(0), \overline{\mu}_B(0)\} \\
&= [min\{\mu_A^L(0), \mu_B^L(0)\}, min\{\mu_A^U(0), \mu_B^U(0)\}] \\
&= [\mu_B^L(0), \mu_B^U(0)] = \overline{\mu}_B(0).
\end{aligned}$$

It follows that

$$\begin{aligned}(\overline{\mu}_A \times \overline{\mu}_B)(x,y) &= rmin\{\overline{\mu}_A(x), \overline{\mu}_B(y)\} \\ &> rmin\{\overline{\mu}_B(0), \overline{\mu}_B(0)\} \\ &= \overline{\mu}_B(0) = (\overline{\mu}_A \times \overline{\mu}_B)(0,0).\end{aligned}$$

This is a contraction.
(iii) Similar to (ii).
(iv) By (i) suppose that $\overline{\mu}_B(0) \geq \overline{\mu}_B(x)$, for all $x \in X$. Form (iii), take $\overline{\mu}_A(0) \geq \overline{\mu}_B(x)$, for all $x \in X$. Then

$$(\overline{\mu}_A \times \overline{\mu}_B)(0,x) = rmin\{\overline{\mu}_A(0), \overline{\mu}_B(x)\} = \overline{\mu}_B(x). \quad (*)$$

Since $A \times B$ is an $i-v$ fuzzy H-ideal, we have

$$\begin{aligned}(\overline{\mu}_A \times \overline{\mu}_B)((x_1,x_2) * (z_1,z_2)) &\geq rmin\{(\overline{\mu}_A \times \overline{\mu}_B)((x_1,x_2) * ((y_1,y_2) * (z_1,z_2))), \\ &\quad (\overline{\mu}_A \times \overline{\mu}_B)(y_1,y_2)\} \\ &= rmin\{(\overline{\mu}_A \times \overline{\mu}_B)(x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)), \\ &\quad (\overline{\mu}_A \times \overline{\mu}_B)(y_1,y_2)\}.\end{aligned}$$

If $x_1 = y_1 = z_1 = 0$, then

$$(\overline{\mu}_A \times \overline{\mu}_B)(0, x_2 * z_2) \geq rmin\{(\overline{\mu}_A \times \overline{\mu}_B)(0, x_2 * (y_2 * z_2)), (\overline{\mu}_A \times \overline{\mu}_B)(0, y_2)\}.$$

By $(*)$, we have

$$\overline{\mu}_B(x_2 * z_2) \geq rmin\{\overline{\mu}_B(x_2 * (y_2 * z_2)), \overline{\mu}_B(y_2)\}.$$

This proves that B is an $i-v$ fuzzy H-ideal of X. Similarly, we can show A is an $i-v$ fuzzy H-ideal for case when $\overline{\mu}_A(0) \geq \overline{\mu}_A(x)$ and $\overline{\mu}_B(0) \geq \overline{\mu}_A(x)$ for all $x \in X$. This gives A is an $i-v$ fuzzy H-ideal of X.

Definition 8. (See also [4])Let $\overline{\mu}_B$ be $i-v$ membership function of each element $x \in X$ to the set B. Then strongest $i-v$ fuzzy relation on X, that is a fuzzy relation $\overline{\mu}_A$ on $\overline{\mu}_B$ and μ_{A_B} whose $i-v$ membership function , of each element $(x,y) \in X \times X$ and defined by

$$\overline{\mu}_{A_B}(x,y) = rmin\{\overline{\mu}_B(x), \overline{\mu}_B(y)\}$$

Definition 9. Let $B = [{\mu^L}_B, {\mu^U}_B]$ be an $i-v$ subset in a set X. Then the strongest $i-v$ fuzzy relation on X that is a $i-v$ A on B is A_B and defined by

$$A_B = [\mu^L_{A_B}, \mu^U_{A_B}].$$

Theorem 14. *Let $B = [{\mu^L}_B, {\mu^U}_B]$ be an $i-v$ subset in a set X and $A_B = [\mu^L_{A_B}, \mu^U_{A_B}]$ be the strongest $i-v$ fuzzy relation on X. Then B is an $i-v$ fuzzy H-ideal of X if and only if A_B is $i-v$ fuzzy H-ideal of $X \times X$.*

Proof. Let B be an $i-v$ fuzzy H-ideal of X. Then

$$\overline{\mu}_{A_B}(0,0) = rmin\{\overline{\mu}_B(0), \overline{\mu}_B(0)\} \geq rmin\{\overline{\mu}_B(x), \overline{\mu}_B(y)\} = \overline{\mu}_{A_B}(x,y)$$

for all $(x,y) \in X \times X$. On the other hand

$$\begin{aligned}\overline{\mu}_{A_B}((x_1,x_2)*(z_1,z_2)) &= \overline{\mu}_{A_B}(x_1*z_1, x_2*z_2)\\ &= rmin\{\overline{\mu}_B(x_1*z_1), \overline{\mu}_B(x_2*z_2)\}\\ &\geq rmin\{rmin\{\overline{\mu}_B(x_1*(y_1*z_1)), \overline{\mu}_B(y_1)\}, rmin\{\overline{\mu}_B(x_2*\\ &\quad (y_2*z_2)), \overline{\mu}_B(y_2)\}\}\\ &= rmin\{rmin\{\overline{\mu}_B(x_1*(y_1*z_1)), \overline{\mu}_B(x_2*(y_2*\\ &\quad z_2))\}, rmin\{\overline{\mu}_B(y_1), \overline{\mu}_B(y_2)\}\}\\ &= rmin\{\overline{\mu}_{A_B}(x_1*(y_1*z_1), x_2*(y_2*z_2)), \overline{\mu}_{A_B}(y_1,y_2)\}\\ &= rmin\{\overline{\mu}_{A_B}((x_1,x_2)*((y_1,y_2)*(z_1,z_2))), \overline{\mu}_{A_B}(y_1,y_2)\}\end{aligned}$$

for all $(x_1,x_2), (y_1,y_2), (z_1,z_2)$ in $X \times X$. Hence A_B is an $i-v$ fuzzy H-ideal of $X \times X$.

Conversely, let A_B be an $i-v$ fuzzy H-ideal of $X \times X$. Then for all $(x,x) \in X \times X$, we have

$$rmin\{\overline{\mu}_B(0), \overline{\mu}_B(0)\} = \overline{\mu}_{A_B}(0,0) \geq \overline{\mu}_{A_B}(x,x) = rmin\{\overline{\mu}_B(x), \overline{\mu}_B(x)\}$$

or $\overline{\mu}_B(0) \geq \overline{\mu}_B(x)$ for all $x \in X$. Now, let $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in X \times X$, then

$$\begin{aligned}rmin\{\overline{\mu}_B(x_1*z_1), \overline{\mu}_B(x_2*z_2)\} &= \overline{\mu}_{A_B}(x_1*z_1, x_2*z_2)\\ &= \overline{\mu}_{A_B}((x_1,x_2)*(z_1,z_2))\\ &\geq rmin\{\overline{\mu}_{A_B}((x_1,x_2)*((y_1,y_2)*(z_1,z_2))), \overline{\mu}_{A_B}(y_1,y_2)\}\\ &= rmin\{\overline{\mu}_{A_B}(x_1*(y_1*z_1), x_2*(y_2*z_2)), \overline{\mu}_{A_B}(y_1,y_2)\}\\ &= rmin\{rmin\{\overline{\mu}_B(x_1*(y_1*z_1)), \overline{\mu}_B(y_1)\}, rmin\{\overline{\mu}_B(x_2*\\ &\quad (y_2*z_2)), \mu_B(y_2)\}\}.\end{aligned}$$

If $x_2 = y_2 = z_2 = 0$, then

$$rmin\{\overline{\mu}_B(x_1*z_1), \overline{\mu}_B(0)\} \geq rmin\{rmin\{\overline{\mu}_B(x_1*(y_1*z_1)), \overline{\mu}_B(y_1)\}, \overline{\mu}_B(0)\}$$

or

$$\overline{\mu}_B(x_1*z_1) \geq rmin\{\overline{\mu}_B(x_1*(y_1*z_1)), \overline{\mu}_B(y_1)\}.$$

Therefore B is an $i-v$ fuzzy H-ideal of X.

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