Fuzzy $H$-Ideals of BCI-Algebras with Interval Valued Membership Functions

A. Kordi$^a$, A. Moussavi$^b$, A. Ahmadi$^a$

$^a$Mathematics and Informatics Research Group
ACECR, Tarbiat Modares University
P. O. Box: 14115-343, Tehran, Iran
kordi.ali@gmail.com

$^b$Department of Mathematics
Tarbiat Modares University
P. O. Box: 14115-343, Tehran, Iran
moussavi_a@modares.ac.ir

Abstract. The purpose of this paper is to define the notion of an interval-valued fuzzy $H$-ideal (briefly, an $i-v$ fuzzy $H$-ideal) of a BCI-algebra. Necessary and sufficient conditions for an $i-v$ fuzzy ideal to be an $i-v$ fuzzy $H$-ideal are stated. A way to make a new $i-v$ fuzzy $H$-ideal from old one is given.

Mathematics Subject Classification: 06F35, 03B52

Keywords: BCI-algebra, $H$-ideal, $i-v$ fuzzy $H$-ideal

1. Preliminaries

The notion of BCK-algebras was proposed by Iami and Is’eki in 1966. In the same year, Is’eki [4] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras. Fuzzy sets were initiated by Zadeh [8]. In [7], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as an $i-v$ fuzzy set. In [7], Zadeh also constructed a method of approximate inference using his $i-v$ fuzzy sets. In [2], Biswas defined interval-valued fuzzy subgroups (i.e., $i-v$ fuzzy subgroups) of Rosenfeld’s...
nature, and investigated some elementary properties. In this paper, using the notion of interval-valued fuzzy set by Zadeh, we introduce the concept of an interval-valued fuzzy BCI-subalgebra (briefly, $i-v$ fuzzy BCI-subalgebra) of a BCI-algebra, and study some of their properties. Using an $i-v$ level set of an $i-v$ fuzzy set, we state a characterization of an $i-v$ fuzzy $H$-ideal of BCI-algebras. We prove that every $i-v$ fuzzy $H$-ideal of a BCI-algebra $X$ can be realized as an $i-v$ level $H$-ideal of an $i-v$ fuzzy $H$-ideal of $X$. In connection with the notion of homomorphism, we study how the images and inverse images of $i-v$ fuzzy $H$-ideal become $i-v$ fuzzy $H$-ideal.

By a BCI-algebra we mean an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the following axioms:

1. $((x * y) * (x * z)) * (z * y) = 0$,
2. $(x * (x * y)) * y = 0$,
3. $x * x = 0$,
4. $x * y = 0$ and $y * x = 0$ imply $x = y$.

for all $x, y, z \in X$. We can define a partial ordering "$\leq$" on $X$ by $x \leq y$ if and only if $x * y = 0$.

The following statements are true in any BCI-algebra $X$:

1.1. $(x * y) * z = (x * z) * y$,
1.2. $x * 0 = x$,
1.3. $(x * z) * (y * z) \leq x * y$,
1.4. $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$,
1.5. $0 * (x * y) = (0 * x) * (0 * y)$,
1.6. $x * (x * (x * y)) = x * y$.

**Definition 1.** A non empty subset $I$ of $X$ is called an ideal of $X$ if it satisfies:

$I_1$ $0 \in I$,
$I_2$ $x * y \in I$ and $y \in I$ imply $x \in I$.

**Definition 2.** A nonempty subset $I$ of $X$ is called an $H$-ideal of $X$ if it satisfies condition $(I_1)$ and

$I_3$ $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$.

Putting $z = 0$ in $(I_3)$ then we can see that every $H$-ideal is an ideal.

**Definition 3.** A fuzzy set $\mu$ in a BCI-algebra $X$ is called a fuzzy $H$-ideal of $X$ if

$F I_1$ $\mu(0) \geq \mu(x)$,
$F I_2$ $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}$.
An interval-valued fuzzy set (briefly, \(i-v\) fuzzy set) \(A\) defined on \(X\) is given by
\[
A = \{(x, [\mu_A^L(x), \mu_A^U(x)]) \mid x \in X\} \quad \forall x \in X
\]
where \(\mu_A^L\) and \(\mu_A^U\) are two fuzzy sets in \(X\) such that \(\mu_A^L \leq \mu_A^U\) for all \(x \in X\).

Let \(\mu_A(x) = [\mu_A^L(x), \mu_A^U(x)]\), \(\forall x \in X\) and let \(D[0, 1]\) denotes the family of all closed subintervals of \([0, 1]\). If \(\mu_A^L(x) = \mu_A^U(x) = c\), where \(0 \leq c \leq 1\), say then we have \(\mu_A(x) = [c, c]\) which we also assume, for the sake of convenience, to belong to \(D[0, 1]\). Thus \(\mu_A(x) \in D[0, 1], \forall x \in X\), and therefore the \(i-v\) fuzzy set \(A\) is given by
\[
A = \{(x, \mu_A(x)) \mid x \in X, \mu_A(x) : X \to D[0, 1]\}.
\]

Now let us define what is known as refined minimum (briefly, \(rmin\)) of two elements in \(D[0, 1]\). We also define the symbols \(\geq\), \(\leq\), and \(=\) in case of two elements in \(D[0, 1]\). Consider two elements \(D_1 := [a_1, b_1]\) and \(D_2 := [a_2, b_2] \in D[0, 1]\). Then
\[
rmin(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]
\]
\[
D_1 \geq D_2 \iff a_1 \geq a_2, b_1 \geq b_2;
\]
and similarly we may have \(D_1 \leq D_2\) and \(D_1 = D_2\).

2. INTERVAL-VALUE FUZZY \(H\)-IDEALS OF BCI-ALGEBRAS

**Definition 4.** An Interval valued fuzzy set \(A\) in BCI-algebra \(X\) is called an interval-valued fuzzy \(H\)-ideal of \(X\) if it satisfies
\[FI_1\] \(\mu_A(0) \geq \mu_A(x)\)
\[FI_2\] \(\mu_A(x \ast z) \geq rmin\{\mu_A(x \ast (y \ast z)), \mu_A(y)\}\).

**Example 1.** Consider a BCI-algebra \(X = \{0, a, b, c\}\) with the following Cayley table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
<td>d</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>e</td>
<td>e</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>c</td>
<td>d</td>
<td>a</td>
<td>a</td>
<td>0</td>
</tr>
<tr>
<td>e</td>
<td>e</td>
<td>c</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

let an \(i-v\) fuzzy set \(A\) defined on \(X\) be given by
\[
\mu_A(x) = \begin{cases} 
[0.4, 0.9] & \text{if } x \in \{0, a\} \\
[0.1, 0.3] & \text{otherwise}
\end{cases}
\]

It is easy to check that \(A\) is an \(i-v\) fuzzy \(H\)-ideal of BCI-algebra \(X\).
**Theorem 1.** Let $A$ be an $i - v$ fuzzy $H$-ideal of $X$. If there is a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} \overline{\pi}_A(x_n) = [1, 1]$ then $\overline{\pi}_A(0) = [1, 1]$.

**Proof.** Since $\overline{\pi}_A(0) \geq \overline{\pi}_A(x)$ for all $x \in X$, we have $\overline{\pi}_A(0) \geq \overline{\pi}_A(x_n)$ for every positive integer $n$. Note that

$$[1, 1] \geq \overline{\pi}_A(0) \geq \lim_{n \to \infty} \overline{\pi}_A(x_n) = [1, 1].$$

Hence $\overline{\pi}_A(0) = [1, 1]$.

**Theorem 2.** An $i - v$ fuzzy set $A = [\mu_A, \mu_A']$ in $X$ is an $i - v$ fuzzy $H$-ideal of $X$ if and only if $\mu_A$ and $\mu_A'$ are fuzzy $H$-ideal of $X$.

**Proof.** Since $\mu_A^L(0) \geq \mu_A^L(x)$ and $\mu_A^U(0) \geq \mu_A^U(x)$, therefore $\overline{\pi}_A(0) \geq \overline{\pi}_A(x)$. Suppose that $\mu_A^L$ and $\mu_A^U$ are fuzzy $H$-ideal of $X$. Let $x, y, z \in X$, then

$$\overline{\pi}_A(x \ast z) = [\mu_A^L(x \ast z), \mu_A^U(x \ast z)]$$

$$\geq \min\{\mu_A^L(x \ast (y \ast z)), \mu_A^L(y)\}, \min\{\mu_A^U(x \ast (y \ast z)), \mu_A^U(y)\}$$

$$= r\min\{[\mu_A^L(x \ast (y \ast z)), \mu_A^U(x \ast (y \ast z))], [\mu_A^L(y), \mu_A^U(y)]\}$$

$$= r\min\{[\overline{\pi}_A(x \ast (y \ast z)), \overline{\pi}_A(y)]\}.$$ 

Hence $A$ is an $i - v$ fuzzy $H$-ideal of $X$.

Conversely, assume that $A$ is an $i - v$ fuzzy $H$-ideal of $X$. For any $x, y \in X$, we have

$$[\mu_A^L(x \ast z), \mu_A^U(x \ast z)] = \overline{\pi}_A(x \ast z)$$

$$\geq \min\{\overline{\pi}_A(x \ast (y \ast z)), \overline{\pi}_A(y)\}$$

$$= r\min\{[\mu_A^L(x \ast (y \ast z)), \mu_A^L(y)]\}, \min\{\mu_A^U(x \ast (y \ast z)), \mu_A^U(y)\}$$

It follows that $\mu_A^L(x \ast z) \geq \min\{\mu_A^L(x \ast (y \ast z)), \mu_A^L(y)\}$ and $\mu_A^U(x \ast z) \geq \min\{\mu_A^U(x \ast (y \ast z)), \mu_A^U(y)\}$. Hence $\mu_A^L$ and $\mu_A^U$ are fuzzy $H$-ideal of $X$.

**Lemma 1.** An $i - v$ fuzzy set $A = [\mu_A, \mu_A']$ in $X$ is an $i - v$ fuzzy ideal of $X$ if and only if $\mu_A$ and $\mu_A'$ are fuzzy ideals of $X$.

**Proof.** Since $\mu_A^L(0) \geq \mu_A^L(x)$ and $\mu_A^U(0) \geq \mu_A^U(x)$, therefore $\overline{\pi}_A(0) \geq \overline{\pi}_A(x)$. Suppose that $\mu_A^L$ and $\mu_A^U$ are fuzzy ideals of $X$. Let $x, y \in X$, then

$$\overline{\pi}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$$

$$\geq \min\{\mu_A^L(x \ast y), \mu_A^L(y)\}, \min\{\mu_A^U(x \ast y), \mu_A^U(y)\}$$

$$= r\min\{[\mu_A^L(x \ast y), \mu_A^U(x \ast y)], [\mu_A^L(y), \mu_A^U(y)]\}$$

$$= r\min\{[\overline{\pi}_A(x \ast y), \overline{\pi}_A(y)]\}.$$ 

Hence $A$ is an $i - v$ fuzzy ideal of $X$.

Conversely, assume that $A$ is an $i - v$ fuzzy ideal of $X$. For any $x, y \in X$, we have
\[ [\mu_A^L(x), \mu_A^U(x)] = \overline{\mu}_A(x) \]
\[ \geq \min\{\overline{\mu}_A(x \ast y), \overline{\mu}_A(y)\} \]
\[ = \min\{[\mu_A^L(x \ast y), \mu_A^L(x \ast y)], [\mu_A^L(y), \mu_A^U(y)]\} \]
\[ = \min\{[\mu_A^L(x \ast y), \mu_A^L(y)], \min\{[\mu_A^L(x \ast y), \mu_A^L(y)]\}\}. \]

It follows that \( \mu_A^L(x) \geq \min\{\mu_A^L(x \ast y), \mu_A^L(y)\} \) and \( \mu_A^U(x) \geq \min\{\mu_A^U(x \ast y), \mu_A^U(y)\} \). Hence \( \mu_A^L \) and \( \mu_A^U \) are fuzzy ideals of \( X \).

**Theorem 3.** Every \( i - v \) fuzzy \( H \)-ideal of a BCI-algebra \( X \) is an \( i - v \) fuzzy ideal.

**Proof.** Let \( A = [\mu_A^L, \mu_A^U] \) be an \( i - v \) fuzzy \( H \)-ideals of \( X \), where \( \mu_A^L \) and \( \mu_A^U \) are two fuzzy \( H \)-ideals of BCI-algebra \( X \). Thus \( \mu_A^L \) and \( \mu_A^U \), by Proposition 1 of [5], are fuzzy ideals of \( X \). Hence by lemma 1, \( A \) is an \( i - v \) fuzzy ideal of \( X \).

**Theorem 4.** Let \( A = [\mu_A^L, \mu_A^U] \) be a \( i - v \) fuzzy set in a BCI-algebra \( X \). Then the following statements are equivalent:

(i) \( A \) is an \( i - v \) fuzzy \( H \)-ideal of \( X \),

(ii) for all \( x, y \in X \), \( \overline{\mu}_A(x \ast y) \geq \overline{\mu}_A(x \ast (0 \ast y)) \),

(iii) for all \( x, y, z \in X \), \( \overline{\mu}_A((x \ast y) \ast z) \geq \overline{\mu}_A(x \ast (y \ast z)) \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( \overline{\mu}_A \) be an \( i - v \) fuzzy \( H \)-ideal of \( X \), then

\[ \overline{\mu}_A(x \ast y) \geq \min\{\overline{\mu}_A(x \ast (0 \ast y)), \overline{\mu}_A(0)\} = \overline{\mu}_A(x \ast (0 \ast y)). \]

Therefore \( \overline{\mu}_A(x \ast y) \geq \overline{\mu}_A(x \ast (0 \ast y)) \) for all \( x, y \in X \).

(ii) \( \Rightarrow \) (iii) For all \( x, y, z \in X \), we have

\[ ((x \ast y) \ast (0 \ast z)) \ast (x \ast (y \ast z)) = ((x \ast y) \ast (x \ast (y \ast z))) \ast (0 \ast z) \leq (y \ast z) \ast (0 \ast z) \]
\[ = (0 \ast z) \ast (0 \ast z) = 0, \]

hence \( (x \ast y) \ast (0 \ast z) \leq x \ast (y \ast z) \). Therefore

\[ \overline{\mu}_A((x \ast y) \ast (0 \ast z)) = \min\{[\mu_A^L((x \ast y) \ast (0 \ast z)), \mu_A^U((x \ast y) \ast (0 \ast z))]\} \]
\[ = \overline{\mu}_A(x \ast (y \ast z)). \]

Thus by (ii), we have \( \overline{\mu}_A((x \ast y) \ast z) \geq \overline{\mu}_A(x \ast (y \ast z)). \)

(iii) \( \Rightarrow \) (ii) For all \( x, y, z \in X \), we have

\[ \overline{\mu}_A(x \ast z) \geq \min\{\overline{\mu}_A((x \ast z) \ast y), \overline{\mu}_A(y)\} \]
\[ = \min\{\overline{\mu}_A((x \ast z) \ast y), \overline{\mu}_A(y)\} \]
\[ \geq \min\{\overline{\mu}_A((x \ast z) \ast y), \overline{\mu}_A(y)\}. \]

Therefore \( \overline{\mu}_A(x \ast z) \geq \min\{\overline{\mu}_A((x \ast z) \ast y), \overline{\mu}_A(y)\} \). Hence \( A \) is an \( i - v \) fuzzy \( H \)-ideal of \( X \).

**Theorem 5.** Let \( A \) be an \( i - v \) fuzzy set in a BCI-algebra \( X \). Then \( A \) is an \( i - v \) fuzzy \( H \)-ideal of \( X \) if and only if the nonempty set

\[ \overline{U}(A; [\delta_1, \delta_2]) := \{x \in X | \overline{\mu}_A(x) \geq [\delta_1, \delta_2]\} \]

is a \( H \)-ideal of \( X \) for every \( [\delta_1, \delta_2] \in D[0, 1] \). We then call \( \overline{U}(A; [\delta_1, \delta_2]) \) the \( i - v \) level \( H \)-ideal of \( X \).
Proof. Assume that $A$ is an $i-v$ fuzzy $H$-ideal of $X$. Since for all $x \in \overline{U}(A; [\delta_1, \delta_2])$ we have $\overline{P}_A(0) \geq \overline{P}_A(x) \geq [\delta_1, \delta_2]$. Therefore $0 \in \overline{U}(A; [\delta_1, \delta_2])$. Now, let $x, y, z \in X$ such that, $x \ast (y \ast z), y \in \overline{U}(A; [\delta_1, \delta_2])$. Then

$$
\overline{P}_A(x \ast z) \geq r\min\{\overline{P}_A(x \ast (y \ast z)), \overline{P}_A(y)\}
\geq r\min\{[\delta_1, \delta_2], [\delta_1, \delta_2]\}
= [\delta_1, \delta_2],
$$

and so $x \ast z \in \overline{U}(A; [\delta_1, \delta_2])$. Thus $\overline{U}(A; [\delta_1, \delta_2])$ is a $H$-ideal of BCI-algebra of $X$.

Conversely, assume that $\overline{U}(A; [\delta_1, \delta_2])(\neq \emptyset)$ is a $H$-ideal of $X$ for every $[\delta_1, \delta_2] \in D[0,1]$. Suppose that there exist $x_0, y_0, z_0 \in X$ such that $\overline{P}_A(x_0 \ast z_0) < r\min\{\overline{P}_A(x_0 \ast (y_0 \ast z_0)), \overline{P}_A(y_0)\}$.

Let $\overline{P}_A(x_0 \ast (y_0 \ast z_0)) = [\beta_1, \beta_2], \overline{P}_A(y_0) = [\beta_3, \beta_4]$, and $\overline{P}_A(x_0 \ast z_0) = [\delta_1, \delta_2]$, then

$$
[\delta_1, \delta_2] < r\min\{[\beta_1, \beta_2], [\beta_3, \beta_4]\}
= \min\{[\beta_1, \beta_3], \min\{\beta_2, \beta_4]\}
$$

Hence $\delta_1 < \min\{\beta_1, \beta_3\}$ and $\delta_2 < \min\{\beta_2, \beta_4\}$. Taking

$$
[\lambda_1, \lambda_2] = \frac{1}{2}(\overline{P}_A(x_0 \ast z_0) + r\min\{\overline{P}_A(x_0 \ast (y_0 \ast z_0)), \overline{P}_A(y_0)\}),
$$

we obtain

$$
[\lambda_1, \lambda_2] = \frac{1}{2}([\delta_1, \delta_2] + \min\{[\beta_1, \beta_3], \min\{\beta_2, \beta_4\}\})
\geq \frac{1}{2}(\delta_1 + \min\{\beta_1, \beta_3\}, \frac{1}{2}(\delta_2 + \min\{\beta_2, \beta_4\})).
$$

It follows that

$$
\min\{\beta_1, \beta_3\} > \beta_1 = \frac{1}{2}(\delta_1 + \min\{\beta_1, \beta_3\}) > \delta_1,
$$

$$
\min\{\beta_2, \beta_4\} > \beta_2 = \frac{1}{2}(\delta_2 + \min\{\beta_2, \beta_4\}) > \delta_2,
$$

so that $[\min\{\beta_1, \beta_3\}, \min\{\beta_2, \beta_4\}] > [\beta_1, \beta_2] = \overline{P}_A(x_0 \ast z_0)$.

Therefore, $x_0 \ast z_0 \not\in \overline{U}(A, [\beta_1, \beta_2])$.

On the other hand,

$$
\overline{P}_A(x_0 \ast (y_0 \ast z_0)) = [\beta_1, \beta_2] \geq [\min\{\beta_1, \beta_3\}, \min\{\beta_2, \beta_4\}] > [\beta_1, \beta_2]
$$

and

$$
\overline{P}_A(y_0) = [\beta_3, \beta_4] \geq [\min\{\beta_1, \beta_3\}, \min\{\beta_2, \beta_4\}] > [\beta_1, \beta_2]
$$

and so $x_0 \ast (y_0 \ast z_0), y_0 \in \overline{U}(A, [\beta_1, \beta_2])$.

But this contradicts the fact that $\overline{U}(A, [\beta_1, \beta_2])$ is a $H$-ideal of $X$. Hence $\overline{P}_A(x \ast z) \geq r\min\{\overline{P}_A(x \ast (y \ast z)), \overline{P}_A(y)\}$, for all $x, y, z \in X$.

Theorem 6. Every $H$-ideal of a BCI-algebra $X$ can be realized as an $i-v$ level $H$-ideal of an $i-v$ fuzzy $H$-ideal of $X$. 
Proof. Let $I$ be a $H$-ideal and $A$ be an $i-v$ fuzzy set on $X$ defined by

$$
\mu_A(x) = \begin{cases} 
[\alpha_1, \alpha_2] & ; \ x \in I \\
[0, 0] & ; \ \text{otherwise}
\end{cases}
$$

where $\alpha_1, \alpha_2 \in (0, 1]$, with $\alpha_1 < \alpha_2$. We show that $A$ is an $i-v$ fuzzy $H$-ideal of $X$. Let $x \ast (y \ast z), y \in I$, then $x \ast z \in I$ and so

$$
\mu_A(x \ast z) = [\alpha_1, \alpha_2] = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = rmin\{\mu_A(x \ast (y \ast z)), \mu_A(y)\}.
$$

If at least one of $x \ast (y \ast z)$ and $y$ is not in $A$, then at least one of $\mu_A(x \ast (y \ast z))$ and $\mu_A(y)$ is 0. Therefore

$$
\mu_A(x \ast z) \geq [0, 0] = rmin\{[0, 0], [0, 0]\} = rmin\{\mu_A(x \ast (y \ast z)), \mu_A(y)\}
$$

This means that $\mu_A$ satisfies $(FI_2)$. On the other hands, since $0 \in I, \mu_A(0) = [\alpha_1, \alpha_2] \geq \mu_A(x)$, for all $x \in X$ and so $\mu_A$ satisfies $(FI_1)$. Thus, $\mu_A$ is an $i-v$ fuzzy $H$-ideal of $X$. It is clear that $\overline{U}(A; [\alpha_1, \alpha_2]) = I$. This completes the proof.

**Theorem 7.** Let $I$ be a subset of a BCI-algebra $X$, such that $0 \in I$ and let $A$ be an $i-v$ fuzzy set on $X$ which is given in the proof of Theorem 5. If $A$ is an $i-v$ fuzzy $H$-ideal of $X$, then $I$ is a $H$-ideal of $X$.

**Proof.** Assume that $A$ is an $i-v$ fuzzy $H$-ideal of $X$. Let $x \ast (y \ast z), y \in I$, then $\mu_A(x \ast (y \ast z)) = [\alpha_1, \alpha_2] = \mu_A(y)$, and so,

$$
\mu_A(x \ast z) \geq rmin\{\mu_A(x \ast (y \ast z)), \mu_A(y)\} = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2].
$$

This implies that $x \ast z \in I$.

**Theorem 8.** If $A$ is an $i-v$ fuzzy $H$-ideal of a BCI-algebra $X$, then the set $X_{\mu_A} := \{x \in X | \mu_A(x) = \mu_A(0)\}$ is a $H$-ideal of $X$.

**Proof.** Let $x \ast (y \ast z), y \in X_{\mu_A}$. Then $\mu_A(y) = \mu_A(0) = \mu_A(x \ast (y \ast z))$, and so

$$
\mu_A(x \ast z) \geq rmin\{\mu_A(x \ast (y \ast z)), \mu_A(y)\} = rmin\{\mu_A(0), \mu_A(0)\} = \mu_A(0).
$$

Therefore $\mu_A(x \ast z) = \mu_A(0)$, that is $x \ast z \in X_{\mu_A}$. Hence $X_{\mu_A}$ is a $H$-ideal of $X$.

**Theorem 9.** For an $i-v$ fuzzy $H$-ideal $A$ of BCI-algebra $X$, the $i-v$ fuzzy set $A^*$ in $X$ defined by $\mu_{A^*}(x) = \mu_A(0 \ast x)$, for all $x \in X$ is an $i-v$ fuzzy $H$-ideal of $X$.

**Proof.** For all $x, y, z \in X$, we have

$$
\mu_{A^*}(x \ast z) = \mu_A(0 \ast (x \ast z)) = \mu_A(0 \ast x) \ast (0 \ast z) \\
\geq rmin\{\mu_A(0 \ast x) \ast ((0 \ast y) \ast (0 \ast z)), \mu_A(0 \ast y)\} = rmin\{\mu_A(0 \ast (x \ast (y \ast z))), \mu_A(0 \ast y)\} = rmin\{\mu_{A^*}(x \ast (y \ast z)), \mu_{A^*}(y)\}.
$$
Therefore $A^*$ is an $i-v$ fuzzy $H$-ideal of $X$.

**Theorem 10.** Let $A$ be an $i-v$ fuzzy ideal of BCI-algebra $X$. If $\mu_A(x \ast y) \geq \mu_A(x)$ for all $x, y \in X$, then $A$ is an $i-v$ fuzzy $H$-ideal of $X$.

**Proof.** Since $A$ is an $i-v$ fuzzy ideal of $X$, by hypothesis we have

$$r_{min}\{\mu_A(x \ast (y \ast z)), \mu_A(y)\} \leq r_{min}\{\mu_A((x \ast z) \ast (y \ast z)), \mu_A(y \ast z)\} \leq \mu_A(x \ast z).$$

For all $x, y, z \in X$. Hence $A$ is an $i-v$ fuzzy $H$-ideal of $X$.

**Definition 5.** ([3]) An $i-v$ fuzzy set $A$ in $X$ is called an interval-valued fuzzy BCI-subalgebra (briefly, $i-v$ fuzzy BCI-subalgebra) of $X$ if

$$\mu_A(x \ast y) \geq r_{min}\{\mu_A(x), \mu_A(y)\} \quad \forall x, y \in X.$$

**Theorem 11.** Every $i-v$ fuzzy $H$-ideal of BCI-algebra $X$ is an $i-v$ fuzzy subalgebra of $X$.

**Proof.** Let $A = [\mu_A^L, \mu_A^U]$ be an $i-v$ fuzzy $H$-ideal of $X$, where $\mu_A^L$ and $\mu_A^U$ are two fuzzy $H$-ideal of BCI-algebra $X$. Thus $\mu_A^L$ and $\mu_A^U$ are fuzzy subalgebra of $X$. Hence by Theorem 3.7 of [3], $A$ is an $i-v$ fuzzy subalgebra of $X$.

### 3. Cartesian product of $i-v$ fuzzy $H$-ideals

**Definition 6.** (see also [1]). A fuzzy relation $A$ on any set $X$ is a fuzzy subset $A$ with a membership function $\Omega_A : X \times X \rightarrow [0, 1]$.

**Lemma 2.** (see also [1]) Let $\mu_A$ and $\mu_B$ be membership function of each $x \in X$ to the $i-v$ subsets $A$ and $B$, respectively. Then $\mu_A \times \mu_B$ is membership function of each element $(x, y) \in X \times X$ to the set $A \times B$ and defined by

$$(\mu_A \times \mu_B)(x, y) = r_{min}\{\mu_A(x), \mu_B(y)\}$$

**Definition 7.** Let $A = [\mu_A^L, \gamma_A^U]$ and $B = [\mu_B^L, \gamma_B^U]$ be two $i-v$ fuzzy subsets in a set $X$. The cartesian product of $A$ and $B$ is defined by

$$A \times B = \{(x, y), \mu_A \times \mu_B\}; \forall (x, y) \in X \times X,$$ where $\mu_A \times \mu_B : X \times X \rightarrow D[0, 1]$.

**Theorem 12.** Let $A = [\mu_A^L, \mu_A^U]$ and $B = [\mu_B^L, \mu_B^U]$ be two $i-v$ fuzzy subsets in a set $X$, then $A \times B$ is an $i-v$ fuzzy $H$-ideal of $X \times X$.

**Proof.** Let $(x, y) \in X \times X$, then by definition

$$(\mu_A \times \mu_B)(0, 0) = r_{min}\{\mu_A(0), \mu_B(0)\} = r_{min}\{[\mu_A^L(0), \mu_A^U(0)], [\mu_B^L(0), \mu_B^U(0)]\}$$
Let \( (x',y',z') = (x \ast (y \ast z)) \), then

\[
(x', y', z') = \min\{\mu_A(x), \mu_B(y)\}, \min\{\mu_A(y), \mu_B(z)\}
\]

\[
\geq \min\{\min\{\mu_A(x \ast y), \mu_B(y)\}, \min\{\mu_A(y \ast z), \mu_B(z)\}\}
\]

Therefore \((FI_2)\) holds.

Now, for all \( x, y, z \in X \), we have

\[
(\mathfrak{p}_A \times \mathfrak{p}_B)((x, \hat{x}) \ast (z, \hat{z})) = (\mathfrak{p}_A \times \mathfrak{p}_B)((x \ast z, \hat{x} \ast \hat{z})
\]

\[
= \min\{\mu_A(x \ast z), \mu_B(\hat{x} \ast \hat{z})\}
\]

\[
\geq \min\{\min\{\mu_A(x \ast (y \ast z)), \mu_B(y)\}, \min\{\mu_A(y \ast z), \mu_B(\hat{y} \ast \hat{z})\}\}
\]

\[
= \min\{\min\{\mu_A(x \ast (y \ast z)), \mu_B(y)\}, \min\{\mu_A(y \ast z), \mu_B(\hat{y} \ast \hat{z})\}\}
\]

Hence \( A \times B \) is an \( i - v \) fuzzy \( H \)-ideal of \( X \times X \).

**Theorem 13.** Let \( A = [\mu_A^L, \gamma_A^U] \) and \( B = [\mu_B^L, \gamma_B^U] \) be two \( i - v \) of a set \( X \). If \( A \times B \) is \( i - v \) fuzzy \( H \)-ideal of \( X \times X \), then

(i) \( \overline{p}_A(0) \geq \overline{p}_A(x) \) or \( \overline{p}_B(0) \geq \overline{p}_B(x) \),

(ii) \( \overline{p}_B(0) \geq \overline{p}_A(0) \) or \( \overline{p}_B(0) \geq \overline{p}_B(x) \),

(iii) \( \overline{p}_A(0) \geq \overline{p}_A(x) \) or \( \overline{p}_A(0) \geq \overline{p}_B(x) \),

(iv) \( A \) or \( B \) is an \( i - v \) fuzzy \( H \)-ideal of \( X \).

**Proof.** (i) Suppose that \( \overline{p}_A(0) < \overline{p}_A(x) \) and \( \overline{p}_B(0) < \overline{p}_B(x) \), then

\[
(\mathfrak{p}_A \times \mathfrak{p}_B)((x, \hat{x}) \ast (z, \hat{z})) = \min\{\mu_A(x), \mu_B(y)\}
\]

\[
= \min\{\min\{\mu_A^L(x), \mu_B^L(y)\}, \min\{\mu_A^U(x), \mu_B^U(y)\}\}
\]

\[
\geq \min\{\min\{\mu_A^L(0), \mu_B^L(0)\}, \min\{\mu_A^U(0), \mu_B^U(0)\}\}
\]

This is a contraction.

(ii) Suppose that \( \overline{p}_B(0) < \overline{p}_A(x) \) and \( \overline{p}_B(0) < \overline{p}_B(x) \), then

\[
(\mathfrak{p}_A \times \mathfrak{p}_B)((x, \hat{x}) \ast (z, \hat{z})) = \min\{\mu_A^L(0), \mu_B^L(0)\}
\]

\[
= \min\{\min\{\mu_A^L(0), \mu_B^L(0)\}, \min\{\mu_A^U(0), \mu_B^U(0)\}\}
\]

\[
= \mu_B(0).
\]
It follows that
\[(\overline{\mu}_A \times \overline{\mu}_B)(x, y) = rmin\{\overline{\mu}_A(x), \overline{\mu}_B(y)\} \geq rmin\{\overline{\mu}_B(0), \overline{\mu}_B(0)\} = \overline{\mu}_B(0) = (\overline{\mu}_A \times \overline{\mu}_B)(0, 0).\]

This is a contraction.

(iii) Similar to (ii).

(iv) By (i) suppose that $\overline{\mu}_B(0) \geq \overline{\mu}_B(x)$, for all $x \in X$. Form (iii), take $\overline{\mu}_A(0) \geq \overline{\mu}_B(x)$, for all $x \in X$. Then
\[(\overline{\mu}_A \times \overline{\mu}_B)(0, x) = rmin\{\overline{\mu}_A(0), \overline{\mu}_B(x)\} = \overline{\mu}_B(x). \quad (*)\]

Since $A \times B$ is an $i-v$ fuzzy $H$-ideal, we have
\[(\overline{\mu}_A \times \overline{\mu}_B)((x_1, x_2) \ast (z_1, z_2)) \geq rmin\{(\overline{\mu}_A \times \overline{\mu}_B)((x_1, x_2) \ast ((y_1, y_2) \ast (z_1, z_2))),\]
\[\quad \{\overline{\mu}_A \times \overline{\mu}_B)(y_1, y_2)\}
\[= rmin\{\overline{\mu}_A \times \overline{\mu}_B)(x_1 \ast (y_1 \ast z_1), x_2 \ast (y_2 \ast z_2)),\]
\[\quad \{\overline{\mu}_A \times \overline{\mu}_B)(y_1, y_2)\}.
\]

If $x_1 = y_1 = z_1 = 0$, then
\[(\overline{\mu}_A \times \overline{\mu}_B)(0, x_2 \ast z_2) \geq rmin\{\overline{\mu}_A \times \overline{\mu}_B)(0, x_2 \ast (y_2 \ast z_2)), (\overline{\mu}_A \times \overline{\mu}_B)(0, y_2)\}.
\]

By (*), we have
\[\overline{\mu}_B(x_2 \ast z_2) \geq rmin\{\overline{\mu}_B(x_2 \ast (y_2 \ast z_2)), \overline{\mu}_B(y_2)\}.
\]

This proves that $B$ is an $i-v$ fuzzy $H$-ideal of $X$. Similarly, we can show $A$ is an $i-v$ fuzzy $H$-ideal for case when $\overline{\mu}_A(0) \geq \overline{\mu}_B(x)$ and $\overline{\mu}_B(0) \geq \overline{\mu}_A(x)$ for all $x \in X$. This gives $A$ is an $i-v$ fuzzy $H$-ideal of $X$.

**Definition 8.** (See also [4]) Let $\overline{\mu}_B$ be $i-v$ membership function of each element $x \in X$ to the set $B$. Then strongest $i-v$ fuzzy relation on $X$, that is a fuzzy relation $\overline{\mu}_A$ on $\overline{\mu}_B$ and $\mu_{AB}$ whose $i-v$ membership function, of each element $(x, y) \in X \times X$ and defined by
\[\overline{\mu}_{AB}(x, y) = rmin\{\overline{\mu}_B(x), \overline{\mu}_B(y)\}\]

**Definition 9.** Let $B = [\mu^L_B, \mu^U_B]$ be an $i-v$ subset in a set $X$. Then the strongest $i-v$ fuzzy relation on $X$ that is $i-v$ A on $B$ is $A_B$ and defined by
\[A_B = [\mu^L_{AB}, \mu^U_{AB}].\]

**Theorem 14.** Let $B = [\mu^L_B, \mu^U_B]$ be an $i-v$ subset in a set $X$ and $A_B = [\mu^L_{AB}, \mu^U_{AB}]$ be the strongest $i-v$ fuzzy relation on $X$. Then $B$ is an $i-v$ fuzzy $H$-ideal of $X$ if and only if $A_B$ is $i-v$ fuzzy $H$-ideal of $X \times X$. 
Proof. Let $B$ be an $i - v$ fuzzy $H$-ideal of $X$. Then

$$\overline{m}_{A_B}(0, 0) = r\min\{\overline{m}_B(0), \overline{m}_B(0)\} \geq r\min\{\overline{m}_B(x), \overline{m}_B(y)\} = \overline{m}_{A_B}(x, y)$$

for all $(x, y) \in X \times X$. On the other hand

$$\overline{m}_{A_B}((x_1, x_2) \ast (z_1, z_2)) = \overline{m}_{A_B}(x_1 \ast z_1, x_2 \ast z_2)
\geq r\min\{r\min\{\overline{m}_B(x_1 \ast (y_1 \ast z_1)), \overline{m}_B(y_1)\}, r\min\{\overline{m}_B(x_2 \ast (y_2 \ast z_2)), \overline{m}_B(y_2)\}\}$$

for all $(x_1, x_2), (y_1, y_2), (z_1, z_2)$ in $X \times X$. Hence $A_B$ is an $i - v$ fuzzy $H$-ideal of $X \times X$.

Conversely, let $A_B$ be an $i - v$ fuzzy $H$-ideal of $X \times X$. Then for all $(x, x) \in X \times X$, we have

$$r\min\{\overline{m}_B(0), \overline{m}_B(0)\} = \overline{m}_{A_B}(0, 0) \geq \overline{m}_{A_B}(x, x) = r\min\{\overline{m}_B(x), \overline{m}_B(x)\}$$

or $\overline{m}_B(0) \geq \overline{m}_B(x)$ for all $x \in X$. Now, let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, then

$$r\min\{\overline{m}_B(x_1 \ast z_1), \overline{m}_B(x_2 \ast z_2)\} = \overline{m}_{A_B}(x_1 \ast z_1, x_2 \ast z_2)
\geq r\min\{r\min\{\overline{m}_B((x_1, x_2) \ast (y_1, y_2)), \overline{m}_B(x_1, x_2)\}, \overline{m}_B((y_1, y_2) \ast (z_1, z_2)), \overline{m}_B(y_1, y_2)\}\}$$

If $x_2 = y_2 = z_2 = 0$, then

$$r\min\{\overline{m}_B(x_1 \ast z_1), \overline{m}_B(0)\} \geq r\min\{r\min\{\overline{m}_B(x_1 \ast (y_1 \ast z_1)), \overline{m}_B(y_1)\}, \overline{m}_B(0)\}$$

or

$$\overline{m}_B(x_1 \ast z_1) \geq r\min\{\overline{m}_B(x_1 \ast (y_1 \ast z_1)), \overline{m}_B(y_1)\}.$$ 

Therefore $B$ is an $i - v$ fuzzy $H$-ideal of $X$.

References


Received: November 19, 2007