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Sums of Prime Numbers, the Zeta Function and the π Number

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Abstract

Let $S_m(n)$ be the sum of the m-th powers of the prime factors in the prime factorization of $n!$.

We prove the following asymptotic formula

$$S_m(n) \sim \frac{\zeta(m+1)}{m+1} \frac{n^{m+1}}{\log n}$$

Where $\zeta(s)$ is the Riemann's Zeta Function.

Let $\sigma_{m,p}(n)$ be the sum of the m-th powers of the different prime divisors of n.

We prove the following asymptotic formula

$$\sum_{i=1}^{n} \sigma_{m,p}(i) \sim \frac{\zeta(m+1)}{m+1} \frac{n^{m+1}}{\log n}$$

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1 Lemmas

The following theorems are well known [1]. We shall use them as lemmas.

Let $\pi(x)$ be the number of primes not exceeding x.

Lemma 1.1 *(Prime number theorem). The following formula holds*

$$\pi(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right)$$

Lemma 1.2 *If $p \leq n$ is a prime, then*

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \dots$$

is the exponent of p appearing in the prime factorization of $n!$.

Let $\Omega(n)$ be the number of prime factors in the prime factorization of n.

Lemma 1.3 *The following formula holds*

$$\Omega(n!) = \sum_{i=1}^{n} \Omega(i) = n \log\log n + o\left(n \log\log n\right)$$

The following lemma is a consequence of the prime number theorem (see [3] or [2]).

Let $s_m(x)$ be the sum of the m-th powers of the primes not exceeding x, where $m = 1, 2, 3, \dots$

Lemma 1.4

$$s_m(x) = \sum_{p \leq x} p^m = \frac{x^{m+1}}{(m+1)\log x} + o\left(\frac{x^{m+1}}{\log x}\right)$$

Finally, we can obtain without difficulty the following lemma.

Lemma 1.5

$$\sum_{j=1}^{\infty} \left(\frac{1}{j^m} - \frac{j}{(j+1)^{m+1}}\right) = \zeta(m+1) \qquad (m = 1, 2, 3, \dots)$$

where $\zeta(s)$ is the Riemann's Zeta Function.

Remark. Note that in the proof of lemma 1.5 (if $m = 1$) there exists the formula

$$\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

2 Main Results

Let $S_m(n)$ be the sum of the m-th powers of the prime factors in the prime factorization of $n!$.

Theorem 2.1 *The following asymptotic formula holds*

$$S_m(n) \sim \frac{\zeta(m+1)}{m+1} \frac{n^{m+1}}{\log n} \tag{1}$$

Proof. Lemma 1.2 gives

$$\begin{aligned} S_m(n) &= \sum_{p\le n} p^m\left(\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\dots\right) = \sum_{p\le\sqrt{n}} p^m\left(\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\dots\right) \\ &+ \sum_{\sqrt{n}<p\le n} p^m\left[\frac{n}{p}\right] \end{aligned} \tag{2}$$

Lemma 1.3 gives

$$\begin{aligned} \sum_{p\le\sqrt{n}} p^m\left(\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\dots\right) &\le n^{\frac{m}{2}}\sum_{p\le n}\left(\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\dots\right) \\ &= n^{\frac{m+2}{2}}\log\log n + o\left(n^{\frac{m+2}{2}}\log\log n\right) \end{aligned} \tag{3}$$

Now, we have

$$\begin{aligned} \sum_{\sqrt{n}<p\le n} p^m\left[\frac{n}{p}\right] &= \sum_{\sqrt{n}<p\le\frac{n}{k+1}} p^m\left[\frac{n}{p}\right] + \sum_{\frac{n}{k+1}<p\le\frac{n}{k}} p^m\left[\frac{n}{p}\right] + \sum_{\frac{n}{k}<p\le\frac{n}{k-1}} p^m\left[\frac{n}{p}\right] \\ &+ \dots + \sum_{\frac{n}{3}<p\le\frac{n}{2}} p^m\left[\frac{n}{p}\right] + \sum_{\frac{n}{2}<p\le n} p^m\left[\frac{n}{p}\right] \end{aligned} \tag{4}$$

Note that

$$\sum_{\frac{n}{j+1}<p\le\frac{n}{j}} p^m\left[\frac{n}{p}\right] = j\sum_{\frac{n}{j+1}<p\le\frac{n}{j}} p^m \tag{5}$$

Lemma 1.4 gives

$$\sum_{\frac{n}{j+1}<p\le\frac{n}{j}} p^m = \frac{1}{m+1}\left(\frac{1}{j^{m+1}}-\frac{1}{(j+1)^{m+1}}\right)\frac{n^{m+1}}{\log n} + o\left(\frac{n^{m+1}}{\log n}\right) \tag{6}$$

Substituting (6) into (5) we find that

$$\sum_{\frac{n}{j+1}<p\le\frac{n}{j}} p^m\left[\frac{n}{p}\right] = \frac{1}{m+1}\left(\frac{1}{j^{m}}-\frac{j}{(j+1)^{m+1}}\right)\frac{n^{m+1}}{\log n} + o\left(\frac{n^{m+1}}{\log n}\right) \tag{7}$$

Substituting (7) into (4) we obtain

$$\begin{aligned} \sum_{\sqrt{n}<p\le n} p^m\left[\frac{n}{p}\right] &= \sum_{\sqrt{n}<p\le\frac{n}{k+1}} p^m\left[\frac{n}{p}\right] \\ &+ \frac{n^{m+1}}{\log n}\frac{1}{m+1}\sum_{j=1}^{k}\left(\frac{1}{j^m}-\frac{j}{(j+1)^{m+1}}\right) + o\left(\frac{n^{m+1}}{\log n}\right) \end{aligned} \tag{8}$$

Equations (2), (3), (8) and lemma 1.5 imply (1). The theorem is thus proved.

Lemma 1.4 and theorem 2.1 imply the following relation between the prime numbers and the Riemann's Zeta Function.

Corollary 2.2 *The following limit holds*

$$\lim_{n\to\infty} \frac{S_m(n)}{s_m(n)} = \zeta(m+1)$$

In particular, if m is odd, we obtain infinite relations between the prime numbers and the π number from a well known Euler's theorem. For example:

$$\lim_{n\to\infty} \frac{S_1(n)}{s_1(n)} = \zeta(2) = \frac{\pi^2}{6} \qquad \lim_{n\to\infty} \frac{S_3(n)}{s_3(n)} = \zeta(4) = \frac{\pi^4}{90}$$

Let $\sigma_{m,p}(n)$ be the sum of the m-th powers of the different prime divisors of n.

Theorem 2.3 *The following asymptotic formula holds*

$$\sum_{i=1}^{n} \sigma_{m,p}(i) \sim \frac{\zeta(m+1)}{m+1} \frac{n^{m+1}}{\log n}$$

Proof. We have

$$\sum_{i=1}^{n} \sigma_{m,p}(i) = \sum_{p\leq n} p^m \left[\frac{n}{p}\right]$$

Now, the proof is the same as in theorem 2.1. Theorem 2.3 is proved.

Corollary 2.4 *The following limit holds*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \sigma_{m,p}(i)}{s_m(n)} = \zeta(m+1)$$

References

[1] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers,* Oxford, 1960.

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