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The Leading Coefficient in the Sylvester's Polynomials

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Abstract

We prove using elementary methods a well known theorem on the leading coefficient in the Sylvester's polynomials.

The previous proofs of this theorem are not elementary. These proofs use the theory of a complex variable (see for example [4] where a not elementary proof of a more restricted theorem is given).

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1 Preliminary Results

The following theorem is well known [3]. We shall use it as a lemma.

Lemma 1.1 *If $M \neq 0$ and r are real numbers then*

$$\sum_{i=0}^{K}(Mi+r)^s = \sum_{i=0}^{s+1} A_{i,s}(MK+r)^i \quad (s \geq 0) \quad (0^0 = 1)$$

Where $A_{s+1,s} = 1/(s+1)M$. Besides, if $K=-1$, $\sum_{i=0}^{s+1} A_{i,s}(MK+r)^i = 0$.

Example 1. We have

$$\sum_{i=0}^{K}(Mi+r)^0 = \frac{1}{M}(MK+r) - \frac{r}{M} + 1$$

$$\begin{aligned}\sum_{i=0}^{K}(Mi+r)^2 &= \frac{1}{3M}(MK+r)^3 + \frac{1}{2}(MK+r)^2 + \frac{M}{6}(MK+r) \\ &+ \left(r^2 - \frac{1}{3M}r^3 - \frac{1}{2}r^2 - \frac{M}{6}r\right)\end{aligned}$$

Let us consider the equation ($n \geq 1$)

$$AnXn + \ldots + A1X1 = M'K + r' \tag{1}$$

where the Ai ($i = 1, 2 \ldots, n$) are positive integers and M' is their least common multiple.

It is well known (see [1], or [2] for an elementary proof) for each integer r' there is a polynomial in K, $D_n(M'K + r')$, which count the number of solutions $(Xn, \ldots, X1)$ to the equation, where the Xi ($i = 1, 2 \ldots, n$) are nonnegative integers (Sylvester's polynomial). This polynomial either is the zero polynomial or a polynomial with rational coefficients of degree $n - 1$.

Example 2. Let us consider the equation

$$4X3 + 3X2 + 2X1 = 12K + 8$$

Then we have (see [2])

$$D_3(12K + 8) = 3K^2 + 7K + 4 \qquad (K = 0, 1, 2, \ldots)$$

For example, if $K = 1$ we obtain $D_3(20) = 14$. That is, there exist 14 solutions to the equation. Since this number is little we can obtain easily these 14 solutions, they are: (0,0,10), (5,0,0), (0,2,7), (0,4,4), (0,6,1), (4,0,2), (3,0,4), (2,0,6), (1,0,8), (2,4,0), (1,2,10), (1,4,2), (2,2,3), (3,2,1).

The following theorem is well known [2], we shall use it as a lemma.

Lemma 1.2 *Let us consider the equation*

$$AnXn + A(n-1)X(n-1) + \ldots + A1X1 = M_nK + r' \tag{2}$$

Where $n \geq 2$, $r' \in \{0, \ldots, M_n - 1\}$, M_n is the least common multiple of An, $A(n-1)$,..., $A1$ and M_{n-1} is the least common multiple of $A(n-1), \ldots, A1$.

If $\alpha = M_n/An$, $\beta = M_n/M_{n-1}$, $r' = An\ a + b$ ($0 \leq b < An$), then

$$\begin{aligned} D_n(M_nK + r') &= \sum_{i=0}^{a}\sum_{S=0}^{K} D_{n-1}\left(M_{n-1}(\beta S) + An\ i + b\right) \\ &+ \sum_{i=a+1}^{\alpha-1}\sum_{S=0}^{K-1} D_{n-1}\left(M_{n-1}(\beta S) + An\ i + b\right) \end{aligned}$$

Where $D_n(M_nK + r')$ is the Sylvester's polynomial that correspond to the equation (2) and the $D_{n-1}(M_{n-1}K + An\ i + b)$ are the Sylvester's polynomials that correspond to the equations

$$A(n-1)X(n-1) + \ldots + A1X1 = M_{n-1}K + An\ i + b$$

Lemma 1.3 *The Sylvester's polynomial that correspond to the equation (1) is the zero polynomial if and only if r' is not a multiple of d, where d is the greatest common divisor of the Ai $(i = 1, 2 \ldots, n)$*

Proof. If r' is not a multiple of d then clearly the equation has not solutions, since the Ai and M' are multiples of d. Therefore the Sylvester's polynomial is zero. On the other hand, if r' is multiple of d, since d is the greatest common divisor of the Ai there are integers $Zn, \ldots, Z1$ such that $AnZn + \ldots + A1Z1 = d$. If we multiply both sides by $s = r'/d$ we obtain $AnZ'n + \ldots + A1Z'1 = r'$, where $Z'i = sZi$. Summing to each $Z'i$ a positive multiple of M' thus that the sum Xi is positive we obtain $AnXn + \ldots + A1X1 = M'K + r'$ for some K. That is, there is a solution for some K. Therefore the Sylvester's polynomial is nonzero. Lemma 1.3 is proved.

Note that if we write the Sylvester's polynomials that correspond to the equation

$$AnXn + \ldots + A1X1 = M'K + r'$$

in the variable $N = M'K + r'$ $(N \geq 0)$, we obtain M' polynomials $D_n(N)$ in N which depend only of the residue class of r' modulo M'. We shall call them, Sylvester's polynomials in N.

Newly, each polynomial $D_n(N)$ either is the zero polynomial or a polynomial with rational coefficients of degree $n - 1$. The number of nonzero polynomials will be M'/d (lemma 1.3).

Example 3. Let us consider the equation of example 2.

$$4X3 + 3X2 + 2X1 = 12K + 8$$

Its Sylvester's polynomial is

$$D_3(12K + 8) = 3K^2 + 7K + 4 \qquad (K = 0, 1, 2, \ldots)$$

The Sylvester's polynomial in N will be

$$D_3(N) = 3\left(\frac{N-8}{12}\right)^2 + 7\left(\frac{N-8}{12}\right) + 4 = \frac{1}{48}N^2 + \frac{1}{4}N + \frac{2}{3}$$

Where $N \equiv 8\ (mod\ 12)$ and $N \geq 0$. Analogously there are others 11 Sylvester's polynomials in N which correspond to the equations

$$4X3 + 3X2 + 2X1 = 12K + r' \qquad (r' = 0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11)$$

Lemma 1.4 *Let $Q(N)$ be the nonzero Sylvester's polynomial that correspond to the equation*

$$AnXn + \ldots + A1X1 = M'K + r'$$

Where $N \equiv r'$ $(mod\ M')$ *and* r' *is multiple of* $d = g.c.d(An, \ldots, A1)$.

Let $P(N)$ *be, the Sylvester's polynomial that correspond to the equation*

$$\frac{An}{d}Xn + \ldots + \frac{A1}{d}X1 = \frac{M'}{d}K + \frac{r'}{d}$$

Where $N \equiv (r'/d)$ $(mod(M'/d))$. *Then* $Q(N) = P(N/d)$.

Proof. Both equations have the same number of solutions.

2 Main Results

Theorem 2.1 *Let us consider the equation (1). If* $n \geq 2$, *the leading coefficient of the* M'/d *nonzero Sylvester's polinomials in* N *is the same, and it is*

$$\lambda = \frac{d}{(n-1)!An \ldots A1}$$

Proof. We proceed by mathematical induction. Assume $n = 2$. Let us consider the equation

$$A2X2 + A1X1 = A1A2K + r'$$

Where $r' = 0, \ldots, A1A2 - 1$ and $g.c.d.(A1, A2) = 1$. Then (lemma 1.2): $\alpha = A1$, $\beta = A2$, $r' = A2\ a + b$ $(0 \leq b < A2)$, and

$$\begin{aligned} D_2(A1A2K + r') &= \sum_{i=0}^{a}\sum_{S=0}^{K} D_1\left(A1(A2S) + A2\ i + b\right) \\ &+ \sum_{i=a+1}^{A1-1}\sum_{S=0}^{K-1} D_1\left(A1(A2S) + A2\ i + b\right) \end{aligned}$$

Where $D_1(A1\ K + A2\ i + b)$ is the Sylvester's polynomial that correspond to the equation $A1X1 = A1\ K + A2\ i + b$. Hence

$$D_1(A1\ K + A2\ i + b) = 1 \quad if\ A2\ i + b \quad is\ multiple\ of\ A1$$

$$D_1(A1\ K + A2\ i + b) = 0 \quad if\ A2\ i + b \quad is\ not\ a\ multiple\ of\ A1$$

Note that the numbers $A2\ i + b$ $(i = 0, \ldots, A1 - 1)$ are mutually incongruent modulo $A1$. Hence, there is only a multiple of $A1$. Consequently

$$D_2(A1A2K + r') = K + 1 \quad or \quad D_2(A1A2K + r') = K$$

That is,

$$D_2(N) = \frac{1}{A1A2}N - \frac{r'}{A1A2} + 1$$

or

$$D_2(N) = \frac{1}{A1A2}N - \frac{r'}{A1A2}$$

If $g.c.d.(A1, A2) = d > 1$ and r' is multiple of d, lemma 1.4 gives

$$D_2(N) = \frac{d^2}{A1A2}\frac{N}{d} - \frac{d^2r'}{dA1A2} + 1 = \frac{d}{A1A2}N - \frac{dr'}{A1A2} + 1$$

or

$$D_2(N) = \frac{d}{A1A2}N - \frac{dr'}{A1A2}$$

Hence, the theorem is true if $n = 2$. Suppose the theorem is true for $n-1 \geq 2$. We shall prove it is also true for n. Let us consider the equation

$$AnXn + \ldots + A1X1 = M_nK + r'$$

Where $r' \in \{0, \ldots, M_n - 1\}$ and $g.c.d.(A1, \ldots, An) = 1$. Then (lemma 1.2)

$$\begin{aligned} D_n(M_nK + r') &= \sum_{i=0}^{a}\sum_{S=0}^{K} D_{n-1}\left(M_{n-1}(\beta S) + Ani + b\right) \\ &+ \sum_{i=a+1}^{\alpha-1}\sum_{S=0}^{K-1} D_{n-1}\left(M_{n-1}(\beta S) + Ani + b\right) \end{aligned} \tag{3}$$

$$\alpha = M_n/An, \quad \beta = M_n/M_{n-1}, \quad r' = An\ a + b \quad (0 \leq b < An) \tag{4}$$

where, from the inductive hypothesis

$$D_{n-1}(N) = \frac{1}{(n-2)!}\frac{d}{A1\ldots A(n-1)}N^{n-2} + \ldots$$

$N \equiv An\ i + b\ (mod\ M_{n-1}),\ An\ i + b \equiv 0\ (mod\ d),\ d = g.c.d.(A1 \ldots A(n-1))$

Now, lemma 1.1 gives

$$\begin{aligned} &\sum_{S=0}^{K} D_{n-1}\left(M_{n-1}\beta S + An\ i + b\right) \\ =\ & \sum_{S=0}^{K} \frac{1}{(n-2)!}\frac{d}{A1\ldots A(n-1)}\left(M_{n-1}\beta S + An\ i + b\right)^{n-2} + \ldots \\ =\ & \frac{1}{(n-2)!}\frac{d}{A1\ldots A(n-1)}\frac{1}{(n-1)M_n}\left(M_nK + An\ i + b\right)^{n-1} + \ldots \\ =\ & \frac{1}{(n-1)!}\frac{d}{M_n}\frac{1}{A1\ldots A(n-1)}\left(N + An\ i + b - r'\right)^{n-1} + \ldots \\ =\ & \frac{1}{(n-1)!}\frac{d}{M_n}\frac{1}{A1\ldots A(n-1)}N^{n-1} + \ldots \end{aligned}$$

Note that $M_{n-1}\beta = M_n$ (see (4)). Similarly

$$\sum_{S=0}^{K-1} D_{n-1}\left(M_{n-1}\beta S + An\ i + b\right) = \frac{1}{(n-1)!}\frac{d}{M_n}\frac{1}{A1\ldots A(n-1)}N^{n-1} + \ldots$$

Since $g.c.d.(An, d) = 1$, in the set $0, 1, \ldots, \alpha - 1 = \frac{M_n}{An} - 1$ there are $\frac{M_n}{dAn}$ multiples of d. On the other hand, the equation $Ani + b \equiv 0\ (mod\ d)$ has an unique solution. Hence, in the set $Ani + b$ $(i = 0, 1, \ldots, \alpha - 1 = \frac{M_n}{An} - 1)$ there are $\frac{M_n}{dAn}$ multiples of d. Then, in equation (3) there are $\frac{M_n}{dAn}$ nonzero polynomials (lemma 1.3). Therefore

$$\begin{aligned} D_n(N) &= \frac{M_n}{dAn}\frac{1}{(n-1)!}\frac{d}{M_n}\frac{1}{A1\ldots A(n-1)}N^{n-1} + \ldots \\ &= \frac{1}{(n-1)!}\frac{1}{A1\ldots An}N^{n-1} \end{aligned}$$

If $g.c.d.(A1, \ldots, An) = d > 1$ and r' is multiple of d, lemma 1.4 gives

$$D_n(N) = \frac{1}{(n-1)!}\frac{d^n}{A1\ldots An}\left(\frac{N}{d}\right)^{n-1} + \ldots = \frac{1}{(n-1)!}\frac{d}{A1\ldots An}N^{n-1} + \ldots$$

The theorem is proved.

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