International Mathematical Forum, 3, 2008, no. 25, 1229 - 1232

Closed Divisibility of Integer Sequences under Summation of Digits

Nassar H. S. Haidar

Department of Mathematics, Suffolk University (USA)
Dakar Campus, Senegal
nhaidar@suffolk.edu

Abstract

We report on integer sequences that preserve, under division by basic decimal numbers, the sum of digits in their terms.The terms are shown to generate an infinite commutative semigroup under multiplication.

Mathematics Subject Classifications: 11B13, 05E20

Keywords: Integer sequences, divisibility, closure under summation, commutative semigroups

Integer sequences formed of elements of the decimal set $\Im = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are generally speaking of much interest in the current era of numerical and computational science and engineering. Double-indexed integer sequences in particular, could quite be relevant in discretized wavelet [1], frame and /or Gabor [2] analysis.

In what follows we shall report two new results: (i) on closed divisibility of double-indexed integer sequences and (ii) on their generation of an infinite commutative semigroup (see e.g.[3]) under the multiplication rule of combination.

Theorem 1. *The double − indexed integer sequence*

$$(a_{mn}) = \left(\sum_{k=0}^{\mathcal{M}_n^m} \alpha_{mnk} 10^{\mathcal{M}_n^m - k}\right) , \tag{1}$$

$\forall\ m \in \Im\backslash\{0\}$, $n \in Z^+ = \{0, 1, 2, 3, ..., \infty\}$ *and* $\alpha_{mnk} \in \Im$, *satisfying*

$$(b_n) = \left(\frac{a_{mn}}{m}\right) = \left(\sum_{k=0}^{\mathcal{N}_n} \beta_{nk} 10^{\mathcal{N}_n - k}\right) , \tag{2}$$

with all $\beta_{nk} \in \Im$, *under the sum − of − digits closure condition* :

$$\left(\sum_{k=0}^{\mathcal{M}_n^m} \alpha_{mnk}\right) = \left(\sum_{k=0}^{\mathcal{N}_n} \beta_{nk}\right), \quad \forall\ m \in \Im\backslash\{0\} \tag{3}$$

is the sequence generated by the multiple of 9 general term

$$a_{mn} = 9\, mn , \tag{4}$$

with

$$\sum_{k=0}^{\mathcal{N}_n} \beta_{nk} = 9\, L, \tag{5}$$

where L *is some positive integer*, $\forall\ m \in \Im\backslash\{0\}$ *and* $n \in Z^+$.

Proof. Obviously when $m = 0$, (3) is not satisfied, which restricts the domain for m to $\Im\backslash\{0\}$. The proof is computational, starting with $m = 1$, at the left end of the $\Im\backslash\{0\}$ set, and ending up with $m = 9$ at its right end.

$a_{1n} = 9n =$ 0, 9, 18, 27, 36, 45, 54, 63, 72, 81, 90,....., 459,.....

$a_{2n} = 18n =$ 0, 18, 36, 54, 72, 90, 108, 126, 144, 162, 180,....., 918,.....

$a_{3n} = 27n =$ 0, 27, 54, 81, 108, 135, 162, 189, 216, 243, 270,....., 1377,.....

until

$a_{9n} = 81n =$ 0, 81, 162, 243, 324, 405, 486, 567, 648, 729, 810,....., 4131,.....

All these sequences happen to correspond to the same

$b_n = 9n =$ 0, 9, 18, 27, 36, 45, 54, 63, 72, 81, 90,....., 459,.....

Clearly the sum of the digits in each term of the above sequences is either 9 or a certain multiple of 9 and are equal to the sum of digits of the corresponding term in b_n. Interestingly also the sum of digits in the terms of b_n show their first change from 9 to 18 only at $n = 51$. Moreover, the sum of digits of a much larger number like $b_{789634} =$ 7,106,706 is only 27 ; and this corresponds to $L = 3$. To prove uniqueness of (4), let us examine some integer sequences formed by alternative multiples of the elements of $\Im\backslash\{0\}$ such as 4 mn or 7 mn . These clearly do not satisfy (3).

Conjecture. *L of* (5) *is ordered by some law depending on the number* 51.

The number 51 appears to be a remarkable figure for the integer sequences with the general term $a_{mn} = 9\ mn$. In fact $51 = 3 \times 17$, where the prime number $17 = 8 + 9$, has 9 probably as the upper limit for $\Im$ or the size of the set $\Im\backslash\{0\}$ and 3 is one of the two unit-noninvolving factors of 9.

Theorem 2. *The terms of the infinite sequences with the general term* $a_{mn} = 9\ mn$ *generate ,under multiplication, an infinite commutative semi-group* $\wp$ over $Z^+ \times \Im\backslash\ \{0\}$.

Proof. Assume $a_{mn} = 9\ mn$, $a_{sn} = 9\ sn$ and $a_{rn} = 9\ rn$ to be any three elements of some set $\wp$.Consider then the binary operation of direct product of any two of its elements $a_{mn} \cdot a_{sn} = ms\ b_n^2 = 9\ (9ms\ n^2)$. Since $9ms\ n^2$ is an integer, then $a_{mn} \cdot a_{sn} \in \wp$. Hence $\wp$ is closed under multiplication ; and this is its first group axiom. Moreover $(a_{mn} \cdot a_{sn}) \cdot a_{rn} = a_{mn} \cdot (a_{sn} \cdot a_{rn}) = 9(81msr\ n^3) \in \wp$ proves the associativity axiom, and that completes the requirements for $\wp$ to be a semigroup. Clearly $\wp$ has neither a unit element nor inverse elements. Commutativity of $\wp$ is evident however from the fact that $a_{mn} \cdot a_{sn} = a_{sn} \cdot a_{mn} = ms\ b_n^2 = 9\ (9ms\ n^2)$.

An additional remarkable property of the infinite commutative semigroup $\wp$ is its possession of a zero element . Indeed since $a_{mn} \cdot a_{s0} = 0 \in \wp$, $\forall m, s, n \in Z^+ \times \Im\backslash\{0\}$, then a_{s0} is that zero.

Acknowledgements. I am grateful to Professor Lewis Shaw, Program Coordinator, Dakar Campus of Suffolk University, for his support to this work.

References

[1] C. K. Chui, Wavelets: A Mathematical Tool for Signal Analysis, SIAM, Philadelphia, 1997.

[2] N. H. S. Haidar, A scale-frequency harmonic series. Integral Transforms & Special Functions, 17 (4) (2006), 275-283.

[3] D. Gorenstein, R. Lyons and R. Solomon.The Classification of the

Finite Simple Groups, AMS, Providence, USA, 1994.

Received: November 7, 2007