

# A Projection Algorithm for the Quasiconvex Feasibility Problem<sup>1</sup>

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## Abstract

In the light of Plastria's lower subdifferential, Eremin's projection algorithms for solving the quasiconvex feasibility problem (QFP), where related quasiconvex functions are locally Lipschitzian, but not necessarily differentiable, are extended. The convergence of the proposed algorithm is shown under some mild conditions.

**Keywords:** feasibility problem, quasiconvex function, projection algorithm, subdifferential

## 1 Introduction

A very common problem in diverse areas of mathematics and engineering technology consists of finding a point in the intersection of convex sets. This problem, which is referred to as the convex feasibility problem, is to find a point  $x \in C \subset R^n$ , such that:

$$x \in C = \{x \in R^n : f_i(x) \leq 0, i = 1, \dots, m\}, \quad (1.1)$$

where  $f_1(x), f_2(x), \dots, f_m(x)$  are convex function that defined on  $R^n$ . If  $f_1(x), f_2(x), \dots, f_m(x)$  are continuous quasiconvex functions, that problem is called quasiconvex feasibility problem (QFP). The convex feasibility problem (CFP), which is a special case of the quasiconvex feasibility problem, where related function  $f_i(x)$  ( $i = 1, \dots, m$ ) was convex. The methods of solving CFP was well-studied in the last decades (see [1-3, 6]), in particular, for example, the cyclic subgradient projections (CSP) [5], parallel subgradient projections

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(PSP) [10], general subgradient projection algorithms with Armijo line search [8] and Eremin's algorithmic scheme [7]. While the functions on the left-hand side of the inequalities are quasiconvex, the situation is much more complicated, because such functions lack separation properties that convex functions have. In [4], using different notions for subdifferentials, Censor and Segal propose a projection algorithm for the QFP, which is based on a class of Eremin's algorithms, and this projection algorithm scheme is suitable for only three cases.

In this paper, a general scheme of projection algorithm for solving the quasiconvex feasibility problem is presented and its convergence, which extend the scope of algorithm convergence in [4], is proved under some mild assumptions.

The rest of the paper is organized as follows. Section 2 presents preliminary materials and discusses several notions for subdifferentials. Section 3 gives the generalization of projection algorithm and shows its convergence. Section 4 gives some conclusions.

## 2 Preliminaries

In this section, we review some definitions and basic results which will be used later on.

**Definition 2.1.** Let  $f : C \rightarrow R$ , where  $C$  is a nonempty convex set in  $R^n$ . The function  $f$  is said to be quasiconvex if, for all  $x, y \in C$ , the following inequality holds:

$$f(\theta x + (1 - \theta)y) \leq \max\{f(x), f(y)\} \quad \forall \theta \in (0, 1). \quad (2.1)$$

Evidently, convex functions are quasiconvex, but the converse is not true (for instance, the function  $\log x$  on  $(0, +\infty)$ ).

**Definition 2.2.** A function  $f : R^n \rightarrow R$  is said to satisfy the locally Lipschitzian condition on a set  $C \subset R^n$  if, for any  $x \in C$ , there exists constants  $\delta, L < \infty$  such that

$$\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\| \quad \forall x_1, x_2 \in B(x, \delta). \quad (2.2)$$

**Definition 2.3.** Given a function  $f$  defined on  $R^n$  and a point  $z$ , the Fenchel-Moreau (FM) subdifferential of  $f$  at  $z$  is defined by

$$\partial^{FM} f(z) = \{t \in R^n \mid \langle t, x - z \rangle \leq f(x) - f(z) \quad \forall x \in R^n\}. \quad (2.3)$$

**Definition 2.4.** Given a function  $f$  defined on  $R^n$  and a point  $z$ , the Plastia (P) lower subdifferential of  $f$  at  $z$  is defined by

$$\partial^P f(z) = \{t \in R^n \mid f(x) < f(z) \Rightarrow \langle t, x - z \rangle \leq f(x) - f(z)\}. \tag{2.4}$$

A function  $f$  is called lower subdifferentiable (lsd) on  $K \subseteq R^n$ , if it admits at least one P-lower subgradient at each point. It is clear that every convex function is lsd, since  $\partial^{FM} f(z) \subseteq \partial^P f(z)$ , but not conversely, as real-valued single variable function  $f(x) = |x|^{1/2}$  shows. Plastia in [9] shows that every Lipschitzian quasiconvex function on  $R^n$  has  $\partial^P f(z) \neq \emptyset$ , for every  $z \in R^n$  and says that  $f$  is boundedly subdifferentiable (blsd) on a set  $K \subseteq R^n$ , if at each point of  $K$  there exists a lower subgradient of  $f$  that not exceeding a constant  $L$  which is called a blsd-bound of  $f$ .

**Theorem 2.1 (see [9]).** Every quasiconvex function  $f$  on  $R^n$  that satisfies a Lipschitzian condition with constant  $L$  is blsd on  $R^n$  with blsd-bound  $L$ .

This theorem guarantees the nonemptiness of P-lower subdifferentials.

**Definition 2.5.** Let  $C \subseteq R^n$  be a closed convex set, let  $d(x)$  be a continuous real-valued function, defined on  $R^n$ , that satisfies  $\{x \mid d(x) \leq 0\} = C$ . Let  $e(x)$  be a vector-valued function that is defined and nowhere equal to zero on  $R^n \setminus C$ . Assume also that  $e(x)$  is bounded on any bounded set. Such a pair of functions  $d(x)$  and  $e(x)$  is said to have the d-e property if for arbitrary  $z \notin C$

$$C \subseteq \Omega = \{x \in R^n \mid \langle e(z), x - z \rangle + d(z) \leq 0\}. \tag{2.5}$$

In [4], the authors proposed the pairs of functions  $d(x)$  and  $e(x)$  are chosen by the following three cases:

- (1)  $d(x) = f_j(x), \quad e(x) = L_j \frac{t^j}{\|t^j\|}$ .
- (2)  $d(x) = \sum_{i \in s(x)} K_i f_i(x)$  if  $s(x) \neq \emptyset$  and zero otherwise,  $e(x) = \sum_{i \in s(x)} K_i L_i \frac{t^i}{\|t^i\|}$ .
- (3)  $d(x) = \sum_{i \in s(x)} f_i^2(x)$  if  $s(x) \neq \emptyset$  and zero otherwise,  $e(x) = \sum_{i \in s(x)} L_i f_i(x) \frac{t^i}{\|t^i\|}$ .

### 3 Projection algorithm with convergence

For the QFP, we first assume the following conditions are satisfied.

- (1) Function  $f_i(x)$  are quasiconvex and Lipschitzian with rank  $L_i$  on  $R^n$ , for all  $i \in \{1, \dots, m\}$ ;
- (2) Problem (1) is consistent, i.e.,  $C \neq \emptyset$ .

**Algorithm 3.1 (see [4], Eremin’s algorithmic scheme).**

**Initialization:**  $x^0 \in R^n$  is arbitrary.

**Iterative step:** Given  $x^k$ , calculate the next iterate  $x^{k+1}$  from

$$x^{k+1} = \begin{cases} x^k - \lambda_k \frac{d(x^k)}{\|e(x^k)\|^2} e(x^k), & \text{if } d(x^k) > 0; \\ x^k, & \text{if } d(x^k) \leq 0, \end{cases} \quad (3.1)$$

where the pair of functions  $d(x)$  and  $e(x)$  are user-chosen functions that have the d-e property.

**Relaxation parameters:**  $\{\lambda_k\}$  are confined to the interval  $\varepsilon_1 \leq \lambda_k \leq 2 - \varepsilon_2$ , for all  $k \geq 0$  with some arbitrary small  $\varepsilon_1, \varepsilon_2 \geq 0$ .

While Censor and Segal in [4] discussed this algorithm scheme only for the pairs of functions  $d(x)$  and  $e(x)$  determined by three cases mentioned above. Now we are able to extend the scope of convergence as the following theorem:

**Theory 3.1.** Let  $\{x^k\}$  be a sequence generated by Algorithm 3.1.  $d(x) = f_i(x)$ ,  $e(x) = g_i(x) \in \partial^P f_i(x)$ ,  $i \in s(x) = \{i \mid f_i(x) > 0\}$ . If the solution set of the QFP is nonempty, then  $\{x^k\}$  converges to a solution of the QFP.

*Proof.* Our proof consists of the following three steps:

Step 1: The pair of functions  $d(x)$  and  $e(x)$  have the d-e property;

Step 2: For every  $x \in C$ ,  $k \geq 1$ , we have  $\|x^{k+1} - x\| \leq \|x^k - x\|$ ;

Step 3:  $\lim_{k \rightarrow \infty} x^k = x^* \in C$ .

We now proceed with the proof of each step.

Step 1: From Definition 2.4 and (2.5), we have

$$\langle e(z), x - z \rangle + d(z) = \langle g_i(z), x - z \rangle + f_i(z) \leq f_i(x) \leq 0.$$

Step 2: If  $s(x^k) = \emptyset$ , then  $d(x^k) = f_i(x^k) \leq 0$ ,  $\forall i = 1, \dots, m$ , the problem is solved. Therefore, assume that  $s(x^k) \neq \emptyset$  for all  $k \geq 0$ . Take some  $x \in C$ , From (3.1), we have

$$\begin{aligned} \|x^{k+1} - x\|^2 &= \|x^{k+1} - x^k + x^k - x\|^2 \\ &= \|x^k - x\|^2 + 2\langle x^{k+1} - x^k, x^k - x \rangle + \|x^{k+1} - x^k\|^2 \\ &= \|x^k - x\|^2 + 2\lambda_k \frac{f_i(x^k)}{\|g_i(x^k)\|^2} \langle g_i(x^k), x - x^k \rangle + \lambda_k^2 \frac{f_i^2(x^k)}{\|g_i(x^k)\|^2} \\ &\leq \|x^k - x\|^2 - 2\lambda_k \frac{f_i(x^k)}{\|g_i(x^k)\|^2} \cdot f_i(x^k) + \lambda_k^2 \frac{f_i^2(x^k)}{\|g_i(x^k)\|^2} \\ &= \|x^k - x\|^2 - \lambda_k(2 - \lambda_k) \frac{f_i^2(x^k)}{\|g_i(x^k)\|^2}, \end{aligned}$$

the fact that  $\varepsilon_1 \leq \lambda_k \leq 2 - \varepsilon_2$ , for all  $k > 0$ , yields

$$\|x^{k+1} - x\|^2 \leq \|x^k - x\|^2 - \varepsilon_1 \varepsilon_2 \frac{f_i^2(x^k)}{\|g_i(x^k)\|^2}, \quad (3.2)$$

then we have

$$\|x^{k+1} - x\| \leq \|x^k - x\|, \quad \forall k \geq 0, \quad x \in C.$$

Step 3: For  $x \in C$  the sequence  $\{\|x^k - x\|\}$  is monotonically decreasing and bounded below, therefore, there exist the limit

$$\lim_{k \rightarrow \infty} \|x^k - x\| = d, \tag{3.3}$$

this implies

$$\lim_{k \rightarrow \infty} \frac{f_i^2(x^k)}{\|g_i(x^k)\|^2} = 0,$$

thus

$$\lim_{k \rightarrow \infty} f_i(x^k) = 0, \quad i \in s(x^k). \tag{3.4}$$

Assume that  $\bar{x}$  is an accumulation point of  $\{x^k\}$  and  $\lim_{k_l \rightarrow \infty} x^{k_l} = \bar{x}$ , where  $\{x^{k_l}\}$  is a subsequence of  $\{x^k\}$ . The main purpose of the remaining part of the proof is to show that  $\bar{x}$  is a solution of QFP. From (3.4) and the continuity of  $f_i(x)$ , we know that  $f_i(\bar{x}) = \lim_{k_l \rightarrow \infty} f_i(x^{k_l}) = 0$ , then  $\bar{x} \in C$ . By (3.3) we showed that  $\lim_{k \rightarrow \infty} \|x^k - x\| = d$ , but now  $\lim_{k_l \rightarrow \infty} \|x^{k_l} - \bar{x}\| = 0$ , thus,  $\lim_{k \rightarrow \infty} x^k = \bar{x}$  and the proof is complete.

#### 4 Numerical test

**Example 4.1.** Let

$$f_1(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2 \leq 0,$$

$$f_2(x_1, x_2) = -x_1 - x_2 - \frac{1}{2}(x_1 + x_2)^2 \leq 0.$$

This problem has a unique solution  $x^* = (1.0000, 1.0000)^T$ . By Algorithm 3.1 with Delphi to compile, we choose initial point  $x^* = (100.0000, 100.0000)^T$ ,  $\lambda_0 = 0.5$  and the pair of functions  $d(x)$  and  $e(x)$  which is different from the three cases [4], we get:

$$\begin{aligned} x^0 &= (100.0000, 100.0000)^T \\ x^1 &= (75.2500, 75.2500)^T \\ x^3 &= (56.6875, 56.6875)^T \\ &\vdots \\ x^{32} &= (1.0099, 1.0099)^T. \end{aligned}$$

## 5 Concluding remarks

In this paper, a generalization of projection algorithm for solving the quasiconvex feasibility problem has been presented. Since the Fenchel-Moreau subdifferential might be empty at some points, then it is inapplicable to quasiconvex functions. By using the notion of Plastria lower subdifferential, we extend the validity of the class of Eremin's algorithms to the QFP, the quasiconvex functions on the left-hand side of the inequalities are not necessarily differentiable, but have to satisfy a Lipschitzian condition. The corresponding convergence properties have been established. Also, in forthcoming papers there are better algorithms for solving the QFP.

## References

- [1] R. Aharoni, A. Berman and Y. Censor, An interior points algorithm for the convex feasibility problem, *Adv. Appl. Math*, 4 (1983), 479-489.
- [2] H.H. Bauschke and J.M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev*, 38 (1996), 367-426.
- [3] Y. Censor, Iterative methods for the convex feasibility problem, in M. Rosenfeld and J. Zaks, (eds.), *Convexity and Graph Theory*, Elsevier Science Publishers, Amsterdam, The Netherland, (1984), 83-91.
- [4] Y. Censor and A. Segal, Algorithms for the quasiconvex feasibility problem, *J. Comput. Appl. Math*, 185 (2006), 34-50.
- [5] Y. Censor and A. Lent, Cyclic subgradient projections, *Math. Programming*, 24 (1982), 233-235.
- [6] A.R. De Pierro and A.N. Iusem, A finitely convergent "row-action" method for the convex feasibility problem, *Math.Optim.Appl*, 17 (1998), 225-235.
- [7] I.I. Eremin, On systems of inequalities with convex functions in the left sides, *Amer. Math. Soc. Trans*, 88 (1970), 67-83.
- [8] L. Li and Y. Gao, Projection Algorithm with Line Search for Solving the Convex Feasibility Problem, *J. Inform. Comput*, (to appear).

- [9] F. Plastria, Lower subdifferentiable functions and their minimization by cutting planes, *J. Optim. Theory Appl*, 46 (1985), 37-53.
- [10] L.T. Santos, A parallel subgradient projections method for the convex feasibility problem, *J. Comput. Appl. Math*, 18 (1987), 307-320.

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