Are Quarks Branes on $E_6$?

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Abstract

In this note we relate Coxeter’s orbifold of $E_6$ with 27 points to the 27 left-handed fermions of the Standard Model. Following Type II String Theory, the Clebsch cubic surface associated with the orbifold shows branes emanating from the 18 vertices carrying the coloured up, down and strange quarks and their anti-quarks. This is believed to be the first time that these branes have been constructed. And hints at a Brane World and quantum gravity. The association of a nucleon with the up and down quarks is also discussed in the light of tetrahedral symmetry that is isomorphic to $E_6$.

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1 Introduction

In several papers de Wet [7,8] has described quarks on a cubic surface without pointing out that they are actually branes which is possibly one of the reasons that they have never been directly observed. The cubic surface arises from the fact that a nucleon, which is not a brane, has a tetrahedral symmetry which will be briefly recapped in Section 2. Then by the Mackay correspondence [13] the binary tetrahedral group is isomorphic to the exceptional Lie group $E_6$ which was used some time ago by Slansky [15] as a base for the Standard Model with 27 left-handed fermions. Namely 18 to describe the up, down and strange quarks and 9 for the leptons (cf. de Wet[8]).

In this note a diagram of the enneagons of $E_6$ (which Coxeter [4] labels 2_21) is redrawn in Fig.1 to show the tritangent planes, which are all equilateral triangles tangent to the cubic surface at three points. There are 18 vertices on the orbifold of Fig.1 which could accommodate Type II strings (or the branes of Fig.3) according to Green, Schwarz and Witten [9] Section 9.5.2. For clarity
Fig. 2 shows a section with a triangle labelled by the uud quarks belonging to a proton. An intertwined triangle carries the anti-quarks \( \overline{u}, \overline{u}, \overline{d} \). The triangles may be transformed into themselves by a rotation \( \omega = \exp 2\pi/3 \) of 120 degrees, so the vertices are equivalent, and the labeling arbitrary. Coxeter[5], Section 12.3 uses the annotation \( \lambda, \mu, \nu \) for the powers of \( \omega \) where \( \lambda, \mu, \nu \) can take independently the values 0,1,2. These are appended to Fig. 2 and show that each apex of an equilateral triangle is simply the previous one multiplied by \( \omega \).

Because \( \omega, \omega^2 \) are roots of the irreducible congruence

\[
x^2 + x + 1 = 0 (mod 2)
\]

we may use marks of the Galois Field \( GF[2^2] \) to restore the full symmetry between congruent triangles i.e. analyse only the plane through uud (cf. Coxeter [4]). In Section 3 we will find the corresponding cubic surface which turns out to be the Clebsch Cubic of Fig. 3 where the branes emanating from the triangular vertices are clearly shown. A much enlarged projection of the 6 vertices has been sketched which mirrors that of Fig. 2 and also includes the anti-particles.

If we rotate through 40 degrees we find congruent vertices labeled by ddu for the neutron and \( \overline{d}, \overline{d}, \overline{u} \) for the anti-neutron. A further rotation of 40 degrees brings us to the strange quarks and anti-quarks.

We may envisage tetrahedra in Fig. 2 by appending an additional vertex O in the center of the Figure which is no longer in the same plane. This will be considered in more detail in the following Sections. In this way triangular symmetry about O leads to tetrahedral symmetry.

In a beautiful paper Higashijima et.al. [10] find the Ricci-flat metric of \( E_6 \) in the form

\[
g_{\alpha\beta}|_{\rho=0} = b^{1/19} \partial_\alpha \partial_\beta \Psi \quad (2)
\]

where \( \Psi \) is the Kaehler potential and \( \partial_\alpha \partial_\beta \Psi \) is the Fubini-Study metric of the complex projective space \( CP^n \) with 18 singularities or vertices, each with a radius of \( b^{1/38} \). Equation (2) should be thought of as a metric over sub-manifolds which in the limit as \( b \to 0 \) become conical singularities in Fig. 2 as in Type II String Theory [9].

\( \rho \) is related to a power of a super-manifold or possibly to a 3-D World-brane as in Fig. 3 and considered by Ramirez and Matos [14]. In this scenario quarks and anti-quarks would appear as Dark Matter generated continuously by singularities on the World-brane possibly as a foam. A universe constructed in the small by Causal (i.e. time dependent) Dynamical Triangulations leading to a theory of Quantum Gravity has in fact been proposed by Ambjorn et.al. [1] where Fig. 4 is a picture of what could be dark matter. Perhaps quarks were the first objects created after the 'Big Bang'. They later united to form protons constituting Dark and Star matter. This could still be occurring and may be the source of cosmic rays.
2 Tetrahedral Symmetry and the Nucleon

The equation
\[ \frac{1}{4} \Psi = (iE_4 \psi_1 + E_{23} \psi_2 + E_{14} \psi_3 + E_{05} \psi_4)e \] is a minimal left ideal of the center D of the Dirac ring describing spin about \( x_3 \) [6]. It has been used to model a nucleon because \( E_{23}, E_{05} \) are rotational operators of spin and isospin through half-angles \( \psi_2, \psi_4 \) while \( E_{14} \) is a parity operator satisfying triality. \( iE_4 \psi_1 \) is the identity operator corresponding to rotations (mod\( 2\pi \)).

The connection with tetrahedral symmetry, and therefore \( E_6 \) has been provided by Barth and Nieto[2] who related \( E_{23}, E_{05} \) and \( E_{14} \) to three pairs of ‘Fix-lines’ in the \( \pm i \) eigenspaces defined by \( E^2_{\mu\nu} = -1 \). The 6 fix-lines are the edges of a fundamental equilateral tetrahedron that may be inscribed in a cube so (3) leads naturally to tetrahedral and cubic symmetry governed by the binary tetrahedral and octahedral quaternion groups \( T_d, O \) and their Lie algebras \( E_6, E_7 \) which are isomorphic by the Mackay correspondence [13]. In this way tetrahedral symmetry imposes \( E_6 \) gauge invariance and the theory of elementary particles is reduced to the study of the exceptional Lie algebras \( E_6 \subset E_7 \subset E_8 \).

Finally we shall show that equation (3) is also a primitive idempotent satisfying the logical relation
\[ x^2 = x \] which George Boole [3] elevated to a law of thought and a fundamental axiom of all philosophy (cf.Ch, III prop.IV). This is because (4), written \( x(1-x)=0 \), expresses the fact that it is not possible for anything to possess and at the same time not to possess that quality. For this reason Aristotle believed (4) to be the source of all the axioms of logic and we may think of (3) as the logical foundation of particle physics. Therefore returning to (3) we find the four primitive idempotents
\[ e_1 = -\frac{i}{4}(E_{03} + E_{12} + E_{45} + E_{16}) \]
\[ e_2 = -\frac{i}{4}(-E_{03} - E_{12} + E_{45} + E_{16}) \]
\[ e_3 = -\frac{i}{4}(-E_{03} + E_{12} - E_{45} + E_{16}) \]
\[ e_4 = -\frac{i}{4}(E_{05} - E_{12} - E_{45} + E_{16}) \]

idempotents satisfying \( e_i e_j = 0 \). Eddingtons E-numbers are mapped into the \( 4 \times 4 \) Dirac matrices by
\[ \gamma_\nu = iE_{\bar{0}\nu}, E_{\mu\nu} = E_{\mu\rho}E_{\rho\nu} = -E_{\nu\mu}, \]
\[ E^2_{\mu\nu} = 1, E_{\mu\nu}E_{\sigma\tau} = E_{\sigma\tau}E_{\mu\nu} = iE_{\chi\rho}, \mu < \nu = 1, \ldots, 5 \]
$E_4$ is the unit matrix while $\psi_2, \psi_3$ and $\psi_4$ are half-angles of rotation. Two more equivalent representations of (3) may be obtained by a cyclic interchange of the indices $1, 2, 3$ that express rotations about $x_2, x_3$

To prove that (3) is also idempotent we set $e = e_1$ and note that $e_1E_{23} = E_{23}e_2$ so that $e_1E_{23}e_1 = E_{23}e_2e_1 = 0$. Similarly $e_1E_{14}e_1 = E_{14}e_4e_1 = 0 = e_1E_{05}e_1 = E_{05}e_3e_1$. Therefore from (3)

$$(\Psi/4)^2 = (\Psi/4)iE_4\psi_1$$

which is idempotent if $iE_4\psi_1 = 1$.

### 3 A Cubic Surface

In Ref. [8] de Wet associated the uu,ud lines of principal triangle of Fig. 2 with the operators $E_{05}, E_{50}$ with isospin $\pm 1/2$ and showed that this recipe would yield the quark charges $1/6 \pm 1/2$. The third line was assigned to the parity operator $E_{14}$ so that a change from a left-handed to a right-handed quark is accompanied by charge conjugation. The 3 remaining skew lines meeting at 0 were labeled by $E_{23}, E_{32}$ with spin $\pm 1/2$ and $E_{41}$.

This section will be devoted to finding the cubic surface shown in Figs. 3, 4 that characterizes the tritangent plane uuud. Coxeter[4] utilizes the surface

$s_3 = \sum_{i=0}^{3} x_i^3 = 0$  \hspace{1cm} (8)

on the plane $\sum_{i=0}^{3} x_i = 0$, but this leads only to the intersection of 3 planes so is not very interesting. Instead we could choose the plane

$z_0 + z_1 + \omega z_2 + \omega^2 z_3 = 0$  \hspace{1cm} (9)

passing through $\omega^\lambda \omega^\mu \omega^\nu$ or $(\lambda \mu \nu) = (012)$. However this involves an extra dimension so we turn to the Clebsch cubic

$S_3 = \sum_{i=1}^{5} x_i^3 = \sum_{i=1}^{5} x_i = 0$  \hspace{1cm} (10)

which may be written

$$81(x^3 + y^3 + z^3) - 189(x^2y + x^2z + y^2x + y^2z + z^2x + z^2y) + 54xyz + 126(xy + yx + zx) - 9(x^2 + y^2 + z^2) - 9(x + y + z) + 1 = 0$$  \hspace{1cm} (11)

by utilizing tetrahedral coordinates for the tritangents (cf. Hunt[11], Section 4.1.3)
Equation (11) is plotted on Figs.3 using a programme due to Oliver Labs[12]. Fig.3 clearly shows large branes emanating from the triangular vertices associated with quarks. The mesh is supposed to map gluon paths or lines of electric flux and there are skew orthogonal lepton jets from the interior of Fig.2, possibly coming the innermost triangle, although this cannot have the properties associated with equation (3) so would not be part of a tetrahedron.

References


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![Diagram of the Coxeter Polytope](image)

*Fig. 1*

Figure 1: The Coxeter Polytope
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Figure 2: Section of Fig.1
Figure 3: The Clebsch Cubic