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Minimal Quasi-Ideals of Generalized Transformation Semigroups

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Abstract

Let X and Y be nonempty sets, $P(X,Y)$ denote the set of all mappings with domain in X and range in Y. By a *generalized transformation semigroup* of X into Y we mean a semigroup $(S(X,Y),\theta)$ where $S(X,Y)$ is a nonempty subset of $P(X,Y)$ and $\theta \in P(Y,X)$ with $\alpha\theta\beta \in S(X,Y)$ for all $\alpha,\beta \in S(X,Y)$ and the operation is $*$ defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha,\beta \in S(X,Y)$. A nonzero quasi-ideal Q of a semigroup S (with or without zero) is said to be *minimal* if Q does not properly contain any nonzero quasi-ideal of S. In this paper, all minimal quasi-ideals on some generalized transformation semigroups are characterized. As consequences, all minimal quasi-deals on some standard transformation semigroups are determined.

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1 Introduction and Preliminaries

For a set X, let P_X, T_X, I_X and G_X denote respectively the partial transformation semigroup on X, the full transformation semigroup on X, the one-to-one partial transformation on X (the symmetric inverse semigroup on X) and the symmetric group on X. For $\alpha \in P_X$, the domain and the range of α will be denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. The *rank* of α is $|\operatorname{ran}\alpha|$, the cardinal number of $\operatorname{ran}\alpha$, and it is denoted by $\operatorname{rank}\alpha$. By a *transformation semigroup* on X we mean a subsemigroup of P_X. Also, we let M_X and E_X denote the subsemigroups of T_X defined respectively by

$$M_X = \{\alpha \in T_X \mid \alpha \text{ is one-to-one}\}, E_X = \{\alpha \in T_X \mid \operatorname{ran}\alpha = X\}.$$

Then M_X $[E_X] = G_X$ if and only if X is finite. For $\alpha \in T_X$, α is said to be *one-to-one* at $x \in X$ if $(x\alpha)\alpha^{-1} = \{x\}$ and we call α *almost one-to-one* if the set $\{x \in X \mid \alpha$ is not one-to-one at $x\}$ is finite. Let AM_X be the set of all almost one-to-one transformations of X. Then $M_X \subseteq AM_X \subseteq T_X$. It is easy to verify that for $\alpha, \beta \in T_X$, $A(\alpha\beta) \subseteq A(\alpha) \cup (A(\beta))\alpha^{-1}$ where $A(\gamma) = \{x \in X \mid \gamma$ is not one-to-one at $x\}$ for every $\gamma \in T_X$. This implies that AM_X is a subsemigroup of T_X. It then follows that AM_X is a subsemigroup of T_X containing M_X. A transformation $\alpha \in T_X$ is said to be *almost onto* if $|X \setminus \operatorname{ran}\alpha| < \infty$. It is easily seen that $X \setminus \operatorname{ran}\alpha\beta \subseteq (X \setminus \operatorname{ran}\beta) \cup (X \setminus \operatorname{ran}\alpha)\beta$ for all $\alpha, \beta \in T_X$. Then the set AE_X of all almost onto transformations of X is a subsemigroup of T_X containing E_X. Note that if X is finite, then $AM_X = AE_X = T_X$.

For an infinite set X, let BL_X and OBL_X denote the subsemigroups of T_X defined respectively by

$$BL_X = \{\alpha \in T_X \mid \alpha \text{ is one-to-one and } X \setminus \operatorname{ran}\alpha \text{ is infinite}\}$$

and

$$OBL_X = \{\alpha \in T_X \mid \alpha \text{ is onto and } (x\alpha)\alpha^{-1} \text{ is infinite for all } x \in X\}.$$

The semigroup BL_X is called a *Baer-Levi semigroup* and the semigroup OBL_X is called an *opposite of Baer-Levi semigroup*. We have that $BL_X \subseteq M_X$ and $OBL_X \subseteq E_X$.

R. P. Sullivan [5] generalized transformation semigroups which are called *generalized transformation semigroups* in this paper as follows: If X and Y are any two sets, let $P(X, Y)$ denote the set of all mappings with domain in X and range in Y, that is,

$$P(X, Y) = \{\alpha : A \to Y \mid A \subseteq X\}.$$

Note that $0 \in P(X, Y)$ where 0 is the empty transformation. A *generalized transformation semigroup* of X into Y is a semigroup $(S(X, Y), \theta)$ where $S(X, Y)$ is a nonempty subset of $P(X, Y)$ and $\theta \in P(Y, X)$ with $\alpha\theta\beta \in S(X, Y)$ for all $\alpha, \beta \in S(X, Y)$ and the operation on $S(X, Y)$ is $*$ defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in S(X, Y)$. Note that in [5], this system was called a *generalized partial transformation semigroup*. The notation $T(X, Y)$ will denote the set $\{\alpha \in P(X, Y) \mid \operatorname{dom}\alpha = X\}$. Then $P(X, X) = P_X$ and $T(X, X) = T_X$. The notations $I(X, Y), G(X, Y), M(X, Y), E(X, Y), AM(X, Y)$ and $AE(X, Y)$ are defined analogously, and we then have $I(X, X) = I_X$, $G(X, X) = G_X$, $M(X, X) = M_X$, $E(X, X) = E_X$, $AM(X, X) = AM_X$ and $AE(X, X) = AE_X$. Moreover, for $\theta \in G_X$, (G_X, θ) is a group having θ^{-1} is its identity. Clearly, the map $\alpha \mapsto \alpha\theta$ is an isomorphism of (M_X, θ) onto M_X and also an isomorphism of (E_X, θ) onto E_X. We note that $AM(X,Y) = T(X,Y)$ if $|X| < \infty$ and $AE(X,Y) = T(X,Y)$ if $|Y| < \infty$. For a nonempty subset A of X and $y \in Y$, let A_y denote the element of $P(X,Y)$ with domain A and range $\{y\}$. A subset $S(X,Y)$ of $P(X,Y)$ is said to *cover* X and Y if for every pair $(x,y) \in X \times Y$, there exists an element $A_y \in S(X,Y)$ for some nonempty subset A of X containing x. Our definitions of A_y and the word "cover" are motivated by the paper written by R. P. Sullivan [6].

Let X and Y be infinite sets. Define

$$BL(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is one-to-one and } Y \setminus \operatorname{ran}\alpha \text{ is infinite}\}$$

and

$$OBL(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is onto and } (x\alpha)\alpha^{-1} \text{ is infinite for all } x \in X\}.$$

Then $BL(X,X) = BL_X$ and $OBL(X,X) = OBL_X$. We have that $BL(X,Y) \subseteq M(X,Y)$ and $OBL(X,Y) \subseteq E(X,Y)$.

A subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$. Quasi-ideals are a generalization of one-sided ideals. Moreover, every quasi-ideal of a semigroup S can be written as the intersection of a left ideal and a right ideal of S ([4], page 85). The notion of quasi-ideal for semigroups was introduced by O. Steinfeld [3] in 1956 and it has been widely studied. Many papers cited in [4], a book written by O. Steinfeld in 1978. For a nonempty subset A of a semigroup S, the *quasi-ideal* $(A)_q$ *of* S *generated by* A is the intersection of all quasi-ideals of S containing A ([4], page 10). The following result is well-known.

Proposition 1.1([4], page 85). *For a nonempty subset A of a semigroup S,*

$$(A)_q = S^1A \cap AS^1 = (SA \cap AS) \cup A.$$

The definitions of minimal quasi-ideals and 0-minimal quasi-ideals of semigroups in [4] are given differently as follows: A quasi-ideal Q of a semigroup S without zero is called a *minimal quasi-ideal* of S if Q does not properly contain any quasi-ideal of S. For a semigroup S with zero, a 0-*minimal quasi-ideal* of S is a nonzero quasi-ideal of S which does not properly contain any nonzero quasi-ideal of S. For our study in this paper, we shall use the word "*minimal quasi-ideal*" to represent both "minimal quasi-ideal" and "0-minimal quasi-ideal" defined above. For convenience, we shall consider S itself a minimal quasi-ideal of S if $|S| = 1$. Some generalized transformation semigroups defined previously contain a zero while some don't. In our context, the word a "nonzero quasi-ideal" of any semigroup S is used to represent a quasi ideal of

S if S has no zero and a quasi-ideal Q of S with $Q \neq \{0\}$ if S has a zero 0. A "nonzero element" of any semigroup is used analogously. It is clearly seen that a nonzero quasi-ideal Q of a semigroup S is minimal if and only if $(x)_q = Q$ for every nonzero element $x \in Q$. The following well-known results relating to the minimality of quasi-ideals of semigroups are due to [4].

Theorem 1.2([4], page 27). *A quasi-ideal Q of a semigroup S without zero is minimal if and only if Q is a subgroup of S.*

Theorem 1.3([4], page 35 and 37). *A minimal quasi-ideal of a semigroup S with zero is either a zero subsemigroup or a subgroup with zero of S. If a quasi-ideal of S is a subgroup with zero of S, then Q is a minimal quasi-ideal of S.*

Y. Kemprasit and the author have characterized minimal quasi-ideals of some generalized linear transformation semigroups in [2]. Our aim of this paper is to characterize minimal quasi-ideals of the generalized transformation semigroup $(S(X,Y),\theta)$ where $\theta \in S(Y,X)$ and $S(X,Y)$ is any of $P(X,Y)$, $T(X,Y)$, $I(X,Y)$, $M(X,Y)$, $E(X,Y)$, $AM(X,Y)$ and $AE(X,Y)$. From Theorem 1.2 and Theorem 1.3, a minimal quasi-ideal of $(S(X,Y),\theta)$ is a subgroup, a zero subsemigroup or a subgroup with zero of $(S(X,Y),\theta)$. We shall also indicate what kinds of minimal quasi-ideals we obtain. Note that 0, the empty transformation may and may not belong to $S(X,Y)$. A trivial minimal quasi-ideal of a semigroup S is obviously a subgroup of S.

In the remainder, let X and Y be any nonempty sets.

2 Lemmas

In this section, we give three lemmas which are used in the next section. These give us some general properties of generalized transformation semigroups. In this section, let $S(X,Y)$ be a nonempty subset of $P(X,Y)$, $\theta \in P(Y,X)$ and $(S(X,Y),\theta)$ a generalized transformation semigroup of X into Y.

Lemma 2.1 *For $\alpha \in S(X,Y)$, if $\operatorname{ran}\alpha \cap \operatorname{dom}\theta = \emptyset$ or $\operatorname{dom}\alpha \cap \operatorname{ran}\theta = \emptyset$, then $(\alpha)_q = \{0,\alpha\}$ in $(S(X,Y),\theta)$.*

Proof. Assume that $\operatorname{ran}\alpha \cap \operatorname{dom}\theta = \emptyset$ or $\operatorname{dom}\alpha \cap \operatorname{ran}\theta = \emptyset$. Then $\alpha\theta = 0$ in P_X or $\theta\alpha = 0$ in P_Y. But $(\alpha)_q = (S(X,Y)\theta\alpha \cap \alpha\theta S(X,Y)) \cup \{\alpha\}$ by Proposition 1.1, so we have $(\alpha)_q = \{0\} \cup \{\alpha\} = \{0,\alpha\}$ in $(S(X,Y),\theta)$. ∎

Lemma 2.2 *If $\alpha \in S(X,Y)$ is such that $\operatorname{rank}\alpha = 1$, then in $(S(X,Y),\theta)$, $(\alpha)_q = \{0,\alpha\}$ if $0 \in S(X,Y)$ and $(\alpha)_q = \{\alpha\}$ if $0 \notin S(X,Y)$.*

Proof. Let β be a nonzero element of $(\alpha)_q$. From Proposition 1.1, $\beta = \alpha$ or $\beta = \gamma\theta\alpha = \alpha\theta\lambda$ for some $\gamma, \lambda \in S(X,Y)$. Assume that $\beta = \gamma\theta\alpha = \alpha\theta\lambda$. Then $\operatorname{ran}\beta = \operatorname{ran}(\gamma\theta\alpha) \subseteq \operatorname{ran}\alpha$. From the fact that $\beta \neq 0$, $\operatorname{rank}\alpha = 1$ and $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$, we have $\operatorname{ran}\beta = \operatorname{ran}\alpha$. Since $0 \neq \beta = \alpha\theta\lambda$, it follows that $\operatorname{ran}\alpha \cap \operatorname{dom}\theta\lambda \neq \emptyset$. But $\operatorname{rank}\alpha = 1$, so $\operatorname{ran}\alpha \subseteq \operatorname{dom}\theta\lambda$. Consequently, $\operatorname{ran}\alpha = \operatorname{ran}\alpha \cap \operatorname{dom}\theta\lambda$, and hence $\operatorname{dom}\alpha = (\operatorname{ran}\alpha)\alpha^{-1} = (\operatorname{ran}\alpha \cap \operatorname{dom}\theta\lambda)\alpha^{-1} = \operatorname{dom}(\alpha\theta\lambda) = \operatorname{dom}\beta$. Now we have $\operatorname{dom}\alpha = \operatorname{dom}\beta$, $\operatorname{ran}\alpha = \operatorname{ran}\beta$ and $\operatorname{rank}\alpha = 1$. This implies that $\beta = \alpha$.

Therefore the lemma is proved. ■

Lemma 2.3 *Assume that $S(X,Y)$ covers X and Y. If $\alpha \in S(X,Y)$ is such that $\operatorname{ran}\alpha \cap \operatorname{dom}\theta \neq \emptyset$, $\operatorname{dom}\alpha \cap \operatorname{ran}\theta \neq \emptyset$ and $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y),\theta)$, then $\operatorname{rank}\alpha = 1$.*

proof Let $x \in \operatorname{dom}\alpha \cap \operatorname{ran}\theta$ and $y \in \operatorname{ran}\alpha \cap \operatorname{dom}\theta$. Then $y'\theta = x$ and $x'\alpha = y$ for some $y' \in \operatorname{dom}\theta$ and $x' \in \operatorname{dom}\alpha$. This implies that $x' \in \operatorname{dom}\alpha\theta$ and $y' \in \operatorname{dom}\theta\alpha$. Since $S(X,Y)$ covers X and Y, there exists a subset A of X such that $x'\alpha\theta \in A$ and $A_{y'} \in S(X,Y)$. Consequently, $\operatorname{ran}(\alpha\theta A_{y'}) = \{y'\}$. Since $y' \in \operatorname{dom}\theta\alpha$, it follows that $\operatorname{ran}(\alpha\theta A_{y'}\theta\alpha) = \{y'\theta\alpha\} = \{x\alpha\}$, so $\operatorname{rank}(\alpha\theta A_{y'}\theta\alpha) = 1$. Thus $\alpha\theta A_{y'}\theta\alpha \neq 0$ and $\alpha\theta A_{y'}\theta\alpha \in S(X,Y)\theta\alpha \cap \alpha\theta S(X,Y) \subseteq (\alpha)_q$ in $(S(X,Y),\theta)$ by Proposition 1.1. But $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y),\theta)$, so $(\alpha\theta A_{y'}\theta\alpha)_q = (\alpha)_q$. Because $\operatorname{rank}(\alpha\theta A_{y'}\theta\alpha) = 1$, by Lemma 2.2, we have $\alpha = \alpha\theta A_{y'}\theta\alpha$, and hence $\operatorname{rank}\alpha = 1$. ■

Lemma 2.4 *The following statements hold.*

(i) $M(X,Y) \neq \emptyset$ and $M(Y,X) \neq \emptyset$ if and only if $|X| = |Y|$.

(ii) $E(X,Y) \neq \emptyset$ and $E(Y,X) \neq \emptyset$ if and only if $|X| = |Y|$.

(iii) If φ is a bijection of X onto Y and $\theta \in M(Y,X)$, then $(M(X,Y),\theta) \cong (M_X, \varphi\theta)$.

(iv) If φ is a bijection of X onto Y and $\theta \in E(Y,X)$, then $(E(X,Y),\theta) \cong (E_X, \varphi\theta)$.

Proof. (i) and (ii) are obvious.

(iii) and (iv). Define $\psi_1 : M(X,Y) \to M_X$ by $\alpha\psi_1 = \alpha\varphi^{-1}$ for every $\alpha \in M(X,Y)$ and define $\psi_2 : E(X,Y) \to E_X$ by $\alpha\psi_2 = \alpha\varphi^{-1}$ for every $\alpha \in E(X,Y)$. Clearly, ψ_1 and ψ_2 are isomorphisms of $(M(X,Y),\theta)$ onto $(M_X,\varphi\theta)$ where $\theta \in M(Y,X)$ and of $(E(X,Y),\theta)$ onto $(E_X,\varphi\theta)$ where $\theta \in E(Y,X)$, respectively. ■

Lemma 2.5 *The following statements hold.*

(i) $AM(X,Y) \neq \emptyset$ and $AM(Y,X) \neq \emptyset$ if and only if either both X and Y are finite or both X and Y are infinite and $|X| = |Y|$.

(ii) $AE(X,Y) \neq \emptyset$ and $AE(Y,X) \neq \emptyset$ if and only if either both X and Y are finite or both X and Y are infinite and $|X| = |Y|$.

Proof. First, let $\alpha \in AM(X,Y)$ and $\beta \in AM(Y,X)$. Then

$$A(\alpha) = \{x \in X \mid |(x\alpha)\alpha^{-1}| > 1\}, A(\beta) = \{y \in Y \mid |(y\beta)\beta^{-1}| > 1\},$$

both $A(\alpha)$ and $A(\beta)$ are finite, $\alpha|_{X\setminus A(\alpha)} : X \setminus A(\alpha) \to Y$ is one-to-one and $\beta|_{Y\setminus A(\beta)} : Y \setminus A(\beta) \to X$ is one-to-one. Thus $|X \setminus A(\alpha)| \leqslant |Y|$ and $|Y \setminus A(\beta)| \leqslant |X|$. If X is finite, then Y is finite since $A(\beta)$ is finite and $|Y \setminus A(\beta)| \leqslant |X|$. Next, assume that X is infinite. Since $A(\alpha)$ is finite, it follows that $|X| = |X \setminus A(\alpha)| \leqslant |Y|$, so Y is infinite. But $A(\beta)$ is finite, thus $|Y| = |Y \setminus A(\beta)| \leqslant |X|$. Consequently, $|X| = |Y|$.

Next, let $\gamma \in AE(X,Y)$ and $\lambda \in AE(Y,X)$. Then $|Y\setminus \operatorname{ran}\gamma| < \infty$ and $|X\setminus \operatorname{ran}\lambda| < \infty$. If X is finite, then $\operatorname{ran}\gamma$ is a finite subset of Y, so Y is finite since $|Y\setminus \operatorname{ran}\gamma| < \infty$. Next, assume that X is infinite. But $|X\setminus \operatorname{ran}\lambda| < \infty$, thus $\operatorname{ran}\lambda$ is infinite, and hence Y must be infinite. Consequently,

$$\begin{aligned}
|X| = |\operatorname{ran}\lambda \cup (X \setminus \operatorname{ran}\lambda)| &= |\operatorname{ran}\lambda| + |X \setminus \operatorname{ran}\lambda| \\
&\leqslant |Y| + |X \setminus \operatorname{ran}\lambda| \\
&= |Y| = |\operatorname{ran}\gamma \cup (Y \setminus \operatorname{ran}\gamma)| \\
&\leqslant |\operatorname{ran}\gamma| + |Y \setminus \operatorname{ran}\gamma| \\
&\leqslant |X| + |Y \setminus \operatorname{ran}\gamma| = |X|,
\end{aligned}$$

and thus $|X| = |Y|$.

If both X and Y are finite, then $AM(X,Y) = T(X,Y) = AE(X,Y)$ and $AM(Y,X) = T(Y,X) = AE(Y,X)$. If $|X| = |Y|$, then there is a bijection φ from X onto Y, so $\varphi \in AM(X,Y) \cap AE(X,Y)$ and $\varphi^{-1} \in AM(Y,X) \cap AE(Y,X)$.

Hence (i) and (ii) of the lemma are proved. ■

Lemma 2.6 *The following statements hold.*
(i) $BL(X,Y) \neq \emptyset$ and $M(Y,X) \neq \emptyset$ if and only if $|X| = |Y|$.
(i) $OBL(X,Y) \neq \emptyset$ and $E(Y,X) \neq \emptyset$ if and only if $|X| = |Y|$.

Proof. Since $BL(X,Y) \subseteq M(X,Y)$ and $OBL(X,Y) \subseteq E(X,Y)$, it follows from Lemma 2.4 that either $BL(X,Y) \neq \emptyset$ and $M(Y,X) \neq \emptyset$ or $OBL(X,Y) \neq \emptyset$ and $E(Y,X) \neq \emptyset$ implies that $|X| = |Y|$.

Assume that $|X| = |Y|$. By Lemma 2.4, $M(Y,X) \neq \emptyset$ and $E(Y,X) \neq \emptyset$. Since Y is infinite, there are subsets Y_1 and Y_2 of Y such that $Y = Y_1 \cup Y_2, Y_1 \cap Y_2 = \emptyset$ and $|Y_1| = |Y_2| = |Y|$. Then $|X| = |Y_1| = |Y_2|$. Let $\alpha : X \to Y_1$ be a bijection. Then α is one-to-one and $Y\setminus \operatorname{ran}\alpha = Y \setminus Y_1 = Y_2$ which is infinite. Hence $\alpha \in BL(X,Y)$. Next, we shall show that $OBL(X,Y) \neq \emptyset$. Since X is infinite, $|X \times X| = |X| = |Y|$. Then there is a bijection $\varphi : X \to X \times X$. Consequently,

$$X = \bigcup_{x\in X} (\{x\} \times X)\varphi^{-1} \text{ which is a disjoint union,} \qquad ((1))$$

$$(\{x\} \times X)\varphi^{-1} \text{ is an infinite subset of } X \text{ for every } x \in X. \qquad ((2))$$

Let ψ be a bijection of X onto Y and define $\beta : X \to Y$ by

$$((\{x\} \times X)\varphi^{-1})\beta = x\psi \text{ for all } x \in X. \qquad ((3))$$

From (1), β is well-defined. Since $\operatorname{ran}\psi = Y$, $\operatorname{ran}\beta = Y$ by (3). Moreover, from (1), for each $x \in X, x \in (\{a\} \times X)\varphi^{-1}$ for some $a \in X$, so from (3),

$$(x\beta)\beta^{-1} = (((\{a\} \times X)\varphi^{-1})\beta)\beta^{-1} = (\{a\} \times X)\varphi^{-1}$$

which implies by (2) that $(x\beta)\beta^{-1}$ is infinite. Hence $\beta \in OBL(X, Y)$.

Therefore the lemma is proved. ■

Lemma 2.7 *If $\theta : Y \to X$ is a bijection, then $(BL(X, Y), \theta) \cong BL_X$, through the map $\alpha \mapsto \alpha\theta$.*

Proof. Let $\alpha \in BL(X, Y)$. Then α is one-to-one and $Y \smallsetminus \operatorname{ran}\alpha$ is infinite. Since $\theta : Y \to X$ is a bijection, $\alpha\theta : X \to X$ is one-to-one and

$$X \smallsetminus \operatorname{ran}\alpha\theta = Y\theta \smallsetminus (\operatorname{ran}\alpha)\theta = (Y \smallsetminus \operatorname{ran}\alpha)\theta.$$

Hence $|X \smallsetminus \operatorname{ran}\alpha\theta| = |Y \smallsetminus \operatorname{ran}\alpha|$ since θ is one-to-one. This shows that $\alpha \mapsto \alpha\theta$ is a map from $BL(X, Y)$ into BL_X. This map is one-to-one since θ is one-to-one. If $\alpha \in BL_X$, then we can show similarly as above that $\alpha\theta^{-1} \in BL(X, Y)$. Also, $(\alpha\theta^{-1})\theta = \alpha$ for every $\alpha \in BL_X$. We therefore deduce that the map $\alpha \mapsto \alpha\theta$ is a bijection of $BL(X, Y)$ onto BL_X. This map is a homomorphism from $(BL(X, Y), \theta)$ onto BL_X since for all $\alpha, \beta \in OBL(X, Y), (\alpha\theta\beta)\theta = (\alpha\theta)(\beta\theta)$. ■

Lemma 2.8 *If $\theta : Y \to X$ is a bijection, then $(OBL(X, Y), \theta) \cong OBL_X$, through the map $\alpha \mapsto \alpha\theta$.*

Proof. If $\alpha \in OBL(X, Y)$, then $\operatorname{ran}\alpha\theta = X$ since $\operatorname{ran}\alpha = Y$ and $\operatorname{ran}\theta = X$, and for $x \in X, (x\alpha\theta)(\alpha\theta)^{-1} = (x\alpha\theta)\theta^{-1}\alpha^{-1} = (x\alpha)\alpha^{-1}$ which is infinite. Hence $\alpha \mapsto \alpha\theta$ is a map from $OBL(X, Y)$ into OBL_X. This map is one-to-one since θ is one-to-one. If $\alpha \in OBL_X$, then we have similarly that $\alpha\theta^{-1} \in OBL(X, Y)$. Also, $(\alpha\theta^{-1})\theta = \alpha$ for every $\alpha \in OBL_X$. If $\alpha, \beta \in OBL(X, Y), (\alpha\theta\beta)\theta = (\alpha\theta)(\beta\theta)$. Hence $\alpha \mapsto \alpha\theta$ is an isomorphism of $(OBL(X, Y), \theta)$ onto OBL_X. ■

Lemma 2.9 *The semigroup BL_X has no minimal quasi-ideal.*

Proof. Let $\alpha \in BL_X$. Then α is one-to-one and $X \smallsetminus \operatorname{ran}\alpha$ is infinite. Since $\alpha^2 \in (\alpha)_q$, $(\alpha^2)_q \subseteq (\alpha)_q$. We claim that $(\alpha^2)_q \subsetneq (\alpha)_q$. Suppose on the contrary that $(\alpha^2)_q = (\alpha)_q$. By Proposition 1.1, $\alpha = \alpha^2$ or $\alpha = \beta\alpha^2$ for some $\beta \in BL_X$. But α is one-to-one, so we have $1_X = \alpha$ or $1_X = \beta\alpha$ which implies that $\operatorname{ran}\alpha = X$. This is contrary to the fact that $X \smallsetminus \operatorname{ran}\alpha$ is infinite. Hence we have the claim. Since $\alpha \in BL_X$ is arbitrary, we deduce that BL_X has no minimal quasi-ideal. ■

Lemma 2.10 *The semigroup OBL_X has no minimal quasi-ideal.*

Proof. Let $\alpha \in OBL_X$. Then $\operatorname{ran}\alpha = X$ and $|(x\alpha)\alpha^{-1}| > 1$ for all $x \in X$ and $(\alpha^2)_q \subseteq (\alpha)_q$. Suppose that $(\alpha^2)_q = (\alpha)_q$. By Proposition 1.1, $\alpha = \alpha^2$ or $\alpha = \alpha^2\beta$ for some $\beta \in OBL_X$. Since α is onto, $1_X = \alpha$ or $1_X = \alpha\beta$. Thus α is one-to-one. This is contrary to the fact that $|(x\alpha)\alpha^{-1}| > 1$ for all $x \in X$. Hence $(\alpha^2)_q \subsetneq (\alpha)_q$. Therefore OBL_X has no minimal quasi-ideal. ■

3 Generalized Semigroups of P_X, T_X and I_X

In this section, we characterize minimal quasi-ideals of the semigroup $(S(X,Y), \theta)$ where $S(X,Y)$ is $P(X,Y), T(X,Y)$ or $I(X,Y)$ and $\theta \in S(Y,X)$. It is clearly seen that all $P(X,Y), T(X,Y)$ and $I(X,Y)$ cover X and Y.

Theorem 3.1 *Let $S(X,Y)$ be $P(X,Y)$ or $I(X,Y)$ and $\theta \in S(Y,X)$. Then for $\alpha \in S(X,Y) \smallsetminus \{0\}$, $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y),\theta)$ if and only if one of the following statements holds.*
(i) $\operatorname{ran}\alpha \cap \operatorname{dom}\theta = \emptyset$.
(ii) $\operatorname{dom}\alpha \cap \operatorname{ran}\theta = \emptyset$.
(iii) $\operatorname{rank}\alpha = 1$.
If this is the case, $(\alpha)_q = \{0,\alpha\}$. If $\alpha\theta\alpha = 0$, then $(\alpha)_q$ is a zero subsemigroup of $(S(X,Y),\theta)$. Otherwise, $(\alpha)_q$ is a subgroup with zero of $(S(X,Y),\theta)$.

Proof. To show sufficiency, first assume that (i) or (ii) holds. Then by Lemma 2.1, $(\alpha)_q = \{0,\alpha\}$ in $(S(X,Y),\theta)$, so $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y),\theta)$. If $\operatorname{rank}\alpha = 1$, by Lemma 2.2, $(\alpha)_q = \{0,\alpha\}$ in $(S(X,Y),\theta)$, so it is a minimal quasi-ideal of $(S(X,Y),\theta)$. Clearly, if $\alpha\theta\alpha = 0$, then $(\alpha)_q$ is a zero subsemigroup of $(S(X,Y),\theta)$. By Theorem 1.3, $(\alpha)_q$ is a subgroup with zero of $(S(X,Y),\theta)$ if $\alpha\theta\alpha \neq 0$.

To show necessity, assume that $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y),\theta)$ and suppose that (i) and (ii) are false. Then $\operatorname{ran}\alpha \cap \operatorname{dom}\theta \neq \emptyset$ and $\operatorname{dom}\alpha \cap \operatorname{ran}\theta \neq \emptyset$. But $S(X,Y)$ covers X and Y, so by Lemma 2.3, we deduce that $\operatorname{rank}\alpha = 1$. Hence (iii) holds.

Therefore the theorem is proved. ■

Theorem 3.2 *Let $\theta \in T(Y,X)$. Then for $\alpha \in T(X,Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(T(X,Y),\theta)$ if and only if $\operatorname{rank}\alpha = 1$.*
If this is the case, $(\alpha)_q = \{\alpha\}$.

Proof. Since $\operatorname{dom}\theta = Y \supseteq \operatorname{ran}\beta$ and $\operatorname{dom}\beta = X \supseteq \operatorname{ran}\theta \neq \emptyset$ for every $\beta \in T(X,Y)$, we have from Lemma 2.3 that if $(\alpha)_q$ is a minimal quasi-ideal of $(T(X,Y),\theta)$, then $\operatorname{rank}\alpha = 1$.

For the converse, assume that $\operatorname{rank}\alpha = 1$. Then $(\alpha)_q = \{\alpha\}$ by Lemma 2.2, so $(\alpha)_q$ is a minimal quasi-ideal of $(T(X,Y),\theta)$. ■

4 Generalized Semigroups of M_X, E_X, AM_X and AE_X

We show that the semigroup $(S(X,Y),\theta)$ where $S(X,Y)$ is any of $M(X,Y)$, $E(X,Y)$, $AM(X,Y)$ or $AE(X,Y)$ and $\theta \in S(Y,X)$ has no minimal quasi-ideals except for the case that X and Y are finite.

Theorem 4.1 *The semigroup $(M(X,Y),\theta)$, where $\theta \in M(Y,X)$, has a minimal quasi-ideal if and only if $|X| = |Y| < \infty$.*

If $|X| = |Y| < \infty$, then $(M(X,Y),\theta)$ is a group, so $M(X,Y)$ is itself a unique minimal quasi-ideal of $(M(X,Y),\theta)$.

Proof. By Lemma 2.4(i), $|X| = |Y|$, and from Lemma 2.4(iii), $(M(X,Y),\theta) \cong (M_X, \varphi\theta)$ where $\varphi : X \to Y$ is a bijection. If $|X| < \infty$, then $M_X = G_X$, so $(M_X, \varphi\theta)$ is a group and hence M_X is a unique minimal quasi-ideal of $(M_X, \varphi\theta)$. Then to prove the theorem, it remains to show that $(M_X, \varphi\theta)$ has no minimal quasi-ideals if X is infinite. Let $\overline{\theta} = \varphi\theta$ and assume that X is infinite. Let $a \in X$. Then $|X| = |X \smallsetminus \{a\}|$ since X is infinite. Then there is a one-to-one map β from X onto $X \smallsetminus \{a\}$, so $\beta \in M_X$. Let $\alpha \in M_X$. From Proposition 1.1, we have that $\alpha\overline{\theta}\beta\overline{\theta}\alpha \in (\alpha)_q$ in $(M_X, \overline{\theta})$, so $(\alpha\overline{\theta}\beta\overline{\theta}\alpha)_q \subseteq (\alpha)_q$. If $\alpha \in (\alpha\overline{\theta}\beta\overline{\theta}\alpha)_q$, then from Proposition 1.1, $\alpha = \alpha\overline{\theta}\beta\overline{\theta}\alpha$ or $\alpha = \gamma\overline{\theta}\alpha\overline{\theta}\beta\overline{\theta}\alpha$ for some $\gamma \in M_X$. Since α is one-to-one, we have $1_X = \alpha\overline{\theta}\beta\overline{\theta}$ or $1_X = \gamma\overline{\theta}\alpha\overline{\theta}\beta\overline{\theta}$ where 1_X is the identity map on X. This implies that $\operatorname{ran}(\beta\overline{\theta}) = X$. Hence

$$X = X\beta\overline{\theta} = (X\beta)\overline{\theta} = (X \smallsetminus \{a\})\overline{\theta},$$

so $a\overline{\theta} = b\overline{\theta}$ for some $b \in X \smallsetminus \{a\}$ which is contrary to that $\overline{\theta} : X \to X$ is one-to-one. This proves that $(\alpha\overline{\theta}\beta\overline{\theta}\alpha)_q \subsetneq (\alpha)_q$. Hence we deduce that for every $\alpha \in M_X$, $(\alpha)_q$ is not a minimal quasi-ideal of $(M_X, \overline{\theta})$. Therefore $(M_X, \overline{\theta})$ has no minimal quasi-ideals.

Therefore the theorem is proved. ■

Theorem 4.2 *The semigroup $(E(X,Y),\theta)$, where $\theta \in E(Y,X)$, has a minimal quasi-ideal if and only if $|X| = |Y| < \infty$.*

If $|X| = |Y| < \infty$, then $(E(X,Y),\theta)$ is a group, so $E(X,Y)$ is itself a unique minimal quasi-ideal of $(E(X,Y),\theta)$.

Proof. It follows from Lemma 2.4(ii) and (iv) that $|X| = |Y|$ and $(E(X,Y),\theta) \cong (E_X, \varphi\theta)$ where $\varphi : X \to Y$ is a bijection. If $|X| < \infty$, then $(E_X, \varphi\theta) = (G_X, \varphi\theta)$ which is a group, and thus E_X is a unique minimal quasi-ideal of $(E_X, \varphi\theta)$. Let $\overline{\theta} = \varphi\theta$. Then to prove the theorem it remains to prove that if X is infinite, then $(E_X, \overline{\theta})$ has no minimal quasi-ideals. Assume that X is infinite. Let a, b and c be distinct elements of X. Then $|X \setminus \{a,b\}| = |X \setminus \{c\}|$. Let ψ be a bijection of $X \setminus \{a,b\}$ onto $X \setminus \{c\}$. Let $\beta : X \to X$ be defined by

$$a\beta = b\beta = c \text{ and } x\beta = x\psi \text{ for all } x \in X \setminus \{a,b\}.$$

Then $\beta \in E_X$ but β is not one-to-one. If $\alpha \in E_X$, then from Proposition 1.1, we have $\alpha\overline{\theta}\beta\overline{\theta}\alpha \in (\alpha)_q$ in $(E_X, \overline{\theta})$, and thus $(\alpha\overline{\theta}\beta\overline{\theta}\alpha)_q \subseteq (\alpha)_q$, suppose that $\alpha \in (\alpha\overline{\theta}\beta\overline{\theta}\alpha)_q$. By Proposition 1.1, $\alpha = \alpha\overline{\theta}\beta\overline{\theta}\alpha$ or $\alpha = \alpha\overline{\theta}\beta\overline{\theta}\alpha\overline{\theta}\gamma$ for some $\gamma \in E_X$. Since $\operatorname{ran}\alpha = X$, it follows that $1_X = \overline{\theta}\beta\overline{\theta}\alpha$ or $1_X = \overline{\theta}\beta\overline{\theta}\alpha\overline{\theta}\gamma$. This implies that $\overline{\theta}$ must be one-to-one, so $\overline{\theta} \in G_X$. Hence $\beta\overline{\theta}\alpha = (\overline{\theta})^{-1}$ or $\beta\overline{\theta}\alpha\overline{\theta}\gamma = (\overline{\theta})^{-1}$ which implies that β must be one-to-one, a contradiction. This proves that $(\alpha\overline{\theta}\beta\overline{\theta}\alpha)_q \subsetneq (\alpha)_q$. Therefore we deduce that for every $\alpha \in E_X$, $(\alpha)_q$ is not a minimal quasi-ideal of $(E_X, \overline{\theta})$. This shows that $(E_X, \overline{\theta})$ has no minimal quasi-ideals.

Hence the theorem is completely proved. ■

Theorem 4.3 *For $\theta \in AM(Y,X)$, the semigroup $(AM(X,Y),\theta)$ has a minimal quasi-ideal if and only if X and Y are finite.*

If X and Y are finite, then for $\alpha \in AM(X,Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(AM(X,Y),\theta)$ if and only if $\operatorname{rank}\alpha = 1$. If this is the case, $(\alpha)_q = \{\alpha\}$.

Proof. First assume that X and Y are finite. Then $AM(X,Y) = T(X,Y)$ and $AM(Y,X) = T(Y,X)$. By Theorem 3.2, for every $\alpha \in AM(X,Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(AM(X,Y),\theta)$ if and only if $\operatorname{rank}\alpha = 1$, and for this case, $(\alpha)_q = \{\alpha\}$.

To prove the converse, assume that X or Y is not finite. By Lemma 2.5(i), X and Y are infinite and $|X| = |Y|$. Let $\alpha \in AM(X,Y)$. Then $\alpha\theta \in AM_X$ which implies that $\alpha\theta : X \setminus A(\alpha\theta) \to X$ is one-to-one. Let a and b be distinct elements of $X \setminus A(\alpha\theta)$. Then $a\alpha\theta \neq b\alpha\theta$, so $a\alpha \neq b\alpha$. Let $c \in Y$. Since $|X \setminus \{a\alpha\theta, b\alpha\theta\}| = |X| = |Y| = |Y \setminus \{c\}|$, there is a bijection φ from $X \setminus \{a\alpha\theta, b\alpha\theta\}$ onto $Y \setminus \{c\}$ and define $\beta : X \to Y$ by

$$x\beta = \begin{cases} c & \text{if } x = a\alpha\theta \text{ or } x = b\alpha\theta, \\ x\varphi & \text{if } x \in X \setminus \{a\alpha\theta, b\alpha\theta\}. \end{cases}$$

Then $A(\beta) = \{a\alpha\theta, b\alpha\theta\}$, so $\beta \in AM(X,Y)$. Now we have $a\alpha\theta\beta\theta\alpha = b\alpha\theta\beta\theta\alpha$ and $a\alpha \neq b\alpha$, so $\alpha \neq \alpha\theta\beta\theta\alpha$. By Proposition 1.1, $\alpha\theta\beta\theta\alpha \in (\alpha)_q$. Hence $(\alpha\theta\beta\theta\alpha)_q \subseteq (\alpha)_q$. Suppose that $\alpha \in (\alpha\theta\beta\theta\alpha)_q$. But $\alpha \neq \alpha\theta\beta\theta\alpha$, so from Proposition 1.1, $\alpha = \alpha\theta\beta\theta\alpha\theta\gamma$ for some $\gamma \in AM(X,Y)$. But $a\alpha\theta\beta\theta\alpha = b\alpha\theta\beta\theta\alpha$, so $a\alpha = (a\alpha\theta\beta\theta\alpha)\theta\gamma = (b\alpha\theta\beta\theta\alpha)\theta\gamma = b\alpha$. This is a contradiction. Hence $(\alpha\theta\beta\theta\alpha)_q \subsetneq (\alpha)_q$ which implies that $(\alpha)_q$ is not a minimal quasi-ideal of $(AM(X,Y),\theta)$.

Therefore the proof is complete. ■

Theorem 4.4 *For $\theta \in AE(Y,X)$, the semigroup $(AE(X,Y),\theta)$ has a minimal quasi-ideal if and only if X and Y are finite.*

If X and Y are finite, then for $\alpha \in AE(X,Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(AE(X,Y),\theta)$ if and only if $\operatorname{rank}\alpha = 1$. If this is the case, $(\alpha)_q = \{\alpha\}$.

Proof. If X and Y are finite, then $AE(X,Y) = T(X,Y)$ and $AE(Y,X) = T(Y,X)$, so by Theorem 3.2, for $\alpha \in AE(X,Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(AE(X,Y),\theta)$ if and only if $\operatorname{rank}\alpha = 1$ and for this case, $(\alpha)_q = \{\alpha\}$.

Conversely, assume that X or Y is not finite. By Lemma 2.5(ii), X and Y are infinite and $|X| = |Y|$. Let $\alpha \in AE(X,Y)$. Then $\alpha\theta \in AE_X$, so $X \smallsetminus \operatorname{ran}\alpha\theta$ is finite. Hence $\operatorname{ran}\alpha\theta$ is infinite. Let $a, b \in X$ be such that $a\alpha\theta \neq b\alpha\theta$. Then $a\alpha \neq b\alpha$ and $|X \smallsetminus \{a\alpha\theta, b\alpha\theta\}| = |Y|$. Let $\varphi : X \smallsetminus \{a\alpha\theta, b\alpha\theta\} \to Y$ be a bijection and define $\beta : X \to Y$ by

$$x\beta = \begin{cases} a\alpha & \text{if } x = a\alpha\theta \text{ or } x = b\alpha\theta, \\ x\varphi & \text{if } x \in X \smallsetminus \{a\alpha\theta, b\alpha\theta\}. \end{cases}$$

Then $\beta \in E(X,Y) \subseteq AE(X,Y)$ and $a\alpha\theta\beta\theta\alpha = b\alpha\theta\beta\theta\alpha$. But $a\alpha \neq b\alpha$, so $\alpha \neq \alpha\theta\beta\theta\alpha$. By Proposition 1.1, $\alpha\theta\beta\theta\alpha \in (\alpha)_q$, so $(\alpha\theta\beta\theta\alpha)_q \subseteq (\alpha)_q$. If $\alpha \in (\alpha\theta\beta\theta\alpha)_q$, then by Proposition 1.1, $\alpha = \alpha\theta\beta\theta\alpha\theta\gamma$ for some $\gamma \in AE(X,Y)$ since $\alpha \neq \alpha\theta\beta\theta\alpha$. This implies that $a\alpha = (a\alpha\theta\beta\theta\alpha)\theta\gamma = (b\alpha\theta\beta\theta\alpha)\theta\gamma = b\alpha$, a contradiction. Consequently, $(\alpha\theta\beta\theta\alpha)_q \subsetneq (\alpha)_q$, so $(\alpha)_q$ is not a minimal quasi-ideal of $(AE(X,Y),\theta)$.

Hence the theorem is proved. ■

5 Generalized Semigroups of BL_X and OBL_X

In this section, we show that the semigroup $(BL(X,Y),\theta)$ where $\theta \in M(Y,X)$ and the semigroup $(OBL(X,Y),\theta)$ where $\theta \in E(Y,X)$ have no minimal quasi-ideals.

Theorem 5.1 *For $\theta \in M(Y,X)$, the semigroup $(BL(X,Y),\theta)$ has no minimal quasi-ideal.*

Proof. Let $\theta \in M(Y,X)$. Then $\theta : Y \to X$ is one-to-one.

Case 1: θ is onto. Then $\theta : Y \to X$ is a bijection. By Lemma 2.7, $(BL(X,Y),\theta) \cong BL_X$. But BL_X has no minimal quasi-ideal by Lemma 2.9, so $(BL(X,Y),\theta)$ has no minimal quasi-ideal.

Case 2: θ is not onto. Let $\alpha \in BL(X,Y)$. Then $\alpha\theta\alpha \in (\alpha)_q$, and so $(\alpha\theta\alpha)_q \subseteq (\alpha)_q$. Suppose that $(\alpha\theta\alpha)_q = (\alpha)_q$. By Proposition 1.1, $\alpha = \alpha\theta\alpha$ or $\alpha = \beta\theta\alpha\theta\alpha$ for some $\beta \in BL(X,Y)$. Since α is one-to-one, $1_X = \alpha\theta$ or $1_X = \beta\theta\alpha\theta$ which implies that θ is onto, a contradiction. This shows that

$(\alpha\theta\alpha)_q \subsetneq (\alpha)_q$. We then deduce that $(BL(X,Y),\theta)$ has no minimal quasi-ideal.

Hence the theorem is proved, as desired. ■

Theorem 5.2 *For $\theta \in E(Y,X)$, the semigroup $(OBL(X,Y),\theta)$ has no minimal quasi-ideal.*

Proof. Let $\theta \in E(Y,X)$. Then $\theta : Y \to X$ is onto.

Case 1: θ is one-to-one. Then $\theta : Y \to X$ is a bijection. By Lemma 2.8, $(OBL(X,Y),\theta) \cong OBL_X$. But from Lemma 2.10, OBL_X has no minimal quasi-ideal, so $(OBL(X,Y),\theta)$ has no minimal quasi-ideal.

Case 2: θ is not one-to-one. Let $\alpha \in OBL(X,Y)$. Then ran $\alpha = Y$ and $(x\alpha)\alpha^{-1}$ is infinite for every $x \in X$. Since $\alpha\theta\alpha \in (\alpha)_q, (\alpha\theta\alpha)_q \subseteq (\alpha)_q$. $(\alpha\theta\alpha)_q = (\alpha)_q$. By Proposition 1.1, $\alpha = \alpha\theta\alpha$ or $\alpha = \alpha\theta\alpha\theta\beta$ for some $\beta \in OBL(X,Y)$. Since α is onto, $1_Y = \theta\alpha$ or $1_X = \theta\alpha\theta\beta$ which implies that θ is one-to-one, a contradiction. This shows that $(\alpha\theta\alpha)_q \subsetneq (\alpha)_q$. We then deduce that $(OBL(X,Y),\theta)$ has no minimal quasi-ideal.

Hence the theorem is proved, as desired. ■

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