Minimal Quasi-Ideals of Generalized Transformation Semigroups

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Abstract

Let $X$ and $Y$ be nonempty sets, $P(X, Y)$ denote the set of all mappings with domain in $X$ and range in $Y$. By a generalized transformation semigroup of $X$ into $Y$ we mean a semigroup $(S(X, Y), \theta)$ where $S(X, Y)$ is a nonempty subset of $P(X, Y)$ and $\theta \in P(Y, X)$ with $\alpha \theta \beta \in S(X, Y)$ for all $\alpha, \beta \in S(X, Y)$ and the operation is defined by $\alpha \ast \beta = \alpha \theta \beta$ for all $\alpha, \beta \in S(X, Y)$. A nonzero quasi-ideal $Q$ of a semigroup $S$ (with or without zero) is said to be minimal if $Q$ does not properly contain any nonzero quasi-ideal of $S$. In this paper, all minimal quasi-ideals on some generalized transformation semigroups are characterized. As a consequence, all minimal quasi-ideals on some standard transformation semigroups are determined.

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1 Introduction and Preliminaries

For a set $X$, let $P_X, T_X, I_X$ and $G_X$ denote respectively the partial transformation semigroup on $X$, the full transformation semigroup on $X$, the one-to-one partial transformation on $X$ (the symmetric inverse semigroup on $X$) and the symmetric group on $X$. For $\alpha \in P_X$, the domain and the range of $\alpha$ will be denoted by $\text{dom} \alpha$ and $\text{ran} \alpha$, respectively. The rank of $\alpha$ is $|\text{ran} \alpha|$, the cardinal number of $\text{ran} \alpha$, and it is denoted by $\text{rank} \alpha$. By a transformation semigroup on $X$ we mean a subsemigroup of $P_X$. Also, we let $M_X$ and $E_X$ denote the subsemigroups of $T_X$ defined respectively by

$$M_X = \{ \alpha \in T_X \mid \alpha \text{ is one-to-one} \}, E_X = \{ \alpha \in T_X \mid \text{ran} \alpha = X \}.$$
Then \( M_X [E_X] = G_X \) if and only if \( X \) is finite. For \( \alpha \in T_X \), \( \alpha \) is said to be one-to-one at \( x \in X \) if \((x\alpha)\alpha^{-1} = \{x\}\) and we call \( \alpha \) almost one-to-one if the set \( \{x \in X \mid \alpha \text{ is not one-to-one at } x\} \) is finite. Let \( AM_X \) be the set of all almost one-to-one transformations of \( X \). Then \( M_X \subseteq AM_X \subseteq T_X \). It is easy to verify that for \( \alpha, \beta \in T_X \), \( A(\alpha\beta) \subseteq A(\alpha) \cup (A(\beta))\alpha^{-1} \) where \( A(\gamma) = \{x \in X \mid \gamma \text{ is not one-to-one at } x\} \) for every \( \gamma \in T_X \). This implies that \( AM_X \) is a subsemigroup of \( T_X \). It then follows that \( AM_X \) is a subsemigroup of \( T_X \) containing \( M_X \). A transformation \( \alpha \in T_X \) is said to be almost onto if \(|X \setminus \text{ran } \alpha| < \infty\). It is easily seen that \( X \setminus \text{ran } \alpha \beta \subseteq (X \setminus \text{ran } \alpha) \beta \) for all \( \alpha, \beta \in T_X \). Then the set \( AE_X \) of all almost onto transformations of \( X \) is a subsemigroup containing \( T_X \) containing \( E_X \). Note that if \( X \) is finite, then \( AM_X = AE_X = T_X \).

For an infinite set \( X \), let \( BL_X \) and \( OBL_X \) denote the subsemigroups of \( T_X \) defined respectively by

\[
BL_X = \{\alpha \in T_X \mid \alpha \text{ is one-to-one and } X \setminus \text{ran } \alpha \text{ is infinite}\}
\]

and

\[
OBL_X = \{\alpha \in T_X \mid \alpha \text{ is onto and } (x\alpha)\alpha^{-1} \text{ is infinite for all } x \in X\}.
\]

The semigroup \( BL_X \) is called a \textit{Baer-Levi semigroup} and the semigroup \( OBL_X \) is called an opposite of \textit{Baer-Levi semigroup}. We have that \( BL_X \subseteq M_X \) and \( OBL_X \subseteq E_X \).

R. P. Sullivan [5] generalized transformation semigroups which are called \textit{generalized transformation semigroups} in this paper as follows: If \( X \) and \( Y \) are any two sets, let \( P(X,Y) \) denote the set of all mappings with domain in \( X \) and range in \( Y \), that is,

\[
P(X,Y) = \{\alpha : A \to Y \mid A \subseteq X\}.
\]

Note that \( 0 \in P(X,Y) \) where \( 0 \) is the empty transformation. A \textit{generalized transformation semigroup} of \( X \) into \( Y \) is a semigroup \((S(X,Y),\theta)\) where \( S(X,Y) \) is a nonempty subset of \( P(X,Y) \) and \( \theta \in P(Y,X) \) with \( \alpha \theta \beta \in S(X,Y) \) for all \( \alpha, \beta \in S(X,Y) \) and the operation on \( S(X,Y) \) is \( * \) defined by \( \alpha * \beta = \alpha \theta \beta \) for all \( \alpha, \beta \in S(X,Y) \). Note that in [5], this system was called a \textit{generalized partial transformation semigroup}. The notation \( T(X,Y) \) will denote the set \( \{\alpha \in P(X,Y) \mid \text{dom } \alpha = X\} \). Then \( P(X,X) = P_X \) and \( T(X,X) = T_X \).

The notations \( I(X,Y), G(X,Y), M(X,Y), E(X,Y), AM(X,Y) \) and \( AE(X,Y) \) are defined analogously, and we then have \( I(X,X) = I_X, G(X,X) = G_X, M(X,X) = M_X, E(X,X) = E_X, AM(X,X) = AM_X \) and \( AE(X,X) = AE_X \).

Moreover, for \( \theta \in G_X \), \((G_X,\theta)\) is a group having \( \theta^{-1} \) is its identity. Clearly, the map \( \alpha \mapsto \alpha \theta \) is an isomorphism of \((M_X,\theta)\) onto \( M_X \) and also an isomorphism
of \((E_X, \theta)\) onto \(E_X\). We note that \(AM(X,Y) = T(X,Y)\) if \(|X| < \infty\) and \(AE(X,Y) = T(X,Y)\) if \(|Y| < \infty\). For a nonempty subset \(A\) of \(X\) and \(y \in Y\), let \(A_y\) denote the element of \(P(X,Y)\) with domain \(A\) and range \(\{y\}\). A subset \(S(X,Y)\) of \(P(X,Y)\) is said to cover \(X\) and \(Y\) if for every pair \((x, y) \in X \times Y\), there exists an element \(A_y \in S(X,Y)\) for some nonempty subset \(A\) of \(X\) containing \(x\). Our definitions of \(A_y\) and the word “cover” are motivated by the paper written by R. P. Sullivan [6].

Let \(X\) and \(Y\) be infinite sets. Define

\[
BL(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is one-to-one and } Y \setminus \text{ran } \alpha \text{ is infinite}\}
\]

and

\[
OBL(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is onto and } (x\alpha)^{-1} \text{ is infinite for all } x \in X\}.
\]

Then \(BL(X,X) = BL_X\) and \(OBL(X,X) = OBL_X\). We have that \(BL(X,Y) \subseteq M(X,Y)\) and \(OBL(X,Y) \subseteq E(X,Y)\).

A subsemigroup \(Q\) of a semigroup \(S\) is called a quasi-ideal of \(S\) if \(SQ \cap QS \subseteq Q\). Quasi-ideals are a generalization of one-sided ideals. Moreover, every quasi-ideal of a semigroup \(S\) can be written as the intersection of a left ideal and a right ideal of \(S\) ([4], page 85). The notion of quasi-ideal for semigroups was introduced by O. Steinfeld [3] in 1956 and it has been widely studied. Many papers cited in [4], a book written by O. Steinfeld in 1978. For a nonempty subset \(A\) of a semigroup \(S\), the quasi-ideal \((A)_q\) of \(S\) generated by \(A\) is the intersection of all quasi-ideals of \(S\) containing \(A\) ([4], page 10). The following result is well-known.

**Proposition 1.1**([4], page 85). For a nonempty subset \(A\) of a semigroup \(S\),

\[
(A)_q = S^1A \cap AS^1 = (SA \cap AS) \cup A.
\]

The definitions of minimal quasi-ideals and 0-minimal quasi-ideals of semigroups in [4] are given differently as follows: A quasi-ideal \(Q\) of a semigroup \(S\) without zero is called a minimal quasi-ideal of \(S\) if \(Q\) does not properly contain any quasi-ideal of \(S\). For a semigroup \(S\) with zero, a 0-minimal quasi-ideal of \(S\) is a nonzero quasi-ideal of \(S\) which does not properly contain any nonzero quasi-ideal of \(S\). For our study in this paper, we shall use the word “minimal quasi-ideal” to represent both “minimal quasi-ideal” and “0-minimal quasi-ideal” defined above. For convenience, we shall consider \(S\) itself a minimal quasi-ideal of \(S\) if \(|S| = 1\). Some generalized transformation semigroups defined previously contain a zero while some don’t. In our context, the word a “nonzero quasi-ideal” of any semigroup \(S\) is used to represent a quasi ideal of
$S$ if $S$ has no zero and a quasi-ideal $Q$ of $S$ with $Q \neq \{0\}$ if $S$ has a zero 0. A “nonzero element” of any semigroup is used analogously. It is clearly seen that a nonzero quasi-ideal $Q$ of a semigroup $S$ is minimal if and only if $(x)_Q = Q$ for every nonzero element $x \in Q$. The following well-known results relating to the minimality of quasi-ideals of semigroups are due to [4].

**Theorem 1.2** ([4], page 27). A quasi-ideal $Q$ of a semigroup $S$ without zero is minimal if and only if $Q$ is a subgroup of $S$.

**Theorem 1.3** ([4], page 35 and 37). A minimal quasi-ideal of a semigroup $S$ with zero is either a zero subsemigroup or a subgroup with zero of $S$. If a quasi-ideal of $S$ is a subgroup with zero of $S$, then $Q$ is a minimal quasi-ideal of $S$.

Y. Kemprasit and the author have characterized minimal quasi-ideals of some generalized linear transformation semigroups in [2]. Our aim of this paper is to characterize minimal quasi-ideals of the generalized transformation semigroup $(S(X,Y),\theta)$ where $\theta \in S(Y,X)$ and $S(X,Y)$ is any of $P(X,Y)$, $T(X,Y)$, $I(X,Y)$, $M(X,Y)$, $E(X,Y)$, $AM(X,Y)$ and $AE(X,Y)$. From Theorem 1.2 and Theorem 1.3, a minimal quasi-ideal of $(S(X,Y),\theta)$ is a subgroup, a zero subsemigroup or a subgroup with zero of $(S(X,Y),\theta)$. We shall also indicate what kinds of minimal quasi-ideals we obtain. Note that 0, the empty transformation may and may not belong to $S(X,Y)$. A trivial minimal quasi-ideal of a semigroup $S$ is obviously a subgroup of $S$.

In the remainder, let $X$ and $Y$ be any nonempty sets.

## 2 Lemmas

In this section, we give three lemmas which are used in the next section. These give us some general properties of generalized transformation semigroups. In this section, let $S(X,Y)$ be a nonempty subset of $P(X,Y)$, $\theta \in P(Y,X)$ and $(S(X,Y),\theta)$ a generalized transformation semigroup of $X$ into $Y$.

**Lemma 2.1** For $\alpha \in S(X,Y)$, if $\text{ran } \alpha \cap \text{dom } \theta = \emptyset$ or $\text{dom } \alpha \cap \text{ran } \theta = \emptyset$, then $(\alpha)_q = \{0, \alpha\}$ in $(S(X,Y),\theta)$.

**Proof.** Assume that $\text{ran } \alpha \cap \text{dom } \theta = \emptyset$ or $\text{dom } \alpha \cap \text{ran } \theta = \emptyset$. Then $\alpha \theta = 0$ in $P_X$ or $\theta \alpha = 0$ in $P_Y$. But $(\alpha)_q = (S(X,Y)\theta \alpha \cap \alpha \theta S(X,Y)) \cup \{\alpha\}$ by Proposition 1.1, so we have $(\alpha)_q = \{0\} \cup \{\alpha\} = \{0, \alpha\}$ in $(S(X,Y),\theta)$. ■

**Lemma 2.2** If $\alpha \in S(X,Y)$ is such that $\text{rank } \alpha = 1$, then in $(S(X,Y),\theta)$, $(\alpha)_q = \{0, \alpha\}$ if 0 $\in S(X,Y)$ and $(\alpha)_q = \{\alpha\}$ if 0 $\notin S(X,Y)$.
Proof. Let $\beta$ be a nonzero element of $(\alpha)_q$. From Proposition 1.1, $\beta = \alpha$ or $\beta = \gamma\theta\alpha = \alpha\theta\lambda$ for some $\gamma, \lambda \in S(X,Y)$. Assume that $\beta = \gamma\theta\alpha = \alpha\theta\lambda$. Then $\text{ran} \beta = \text{ran} (\gamma\theta\alpha) \subseteq \text{ran} \alpha$. From the fact that $\beta \neq 0$, $\text{rank} \alpha = 1$ and $\text{ran} \beta \subseteq \text{ran} \alpha$, we have $\text{ran} \beta = \text{ran} \alpha$. Since $0 \neq \beta = \alpha\theta\lambda$, it follows that $\text{ran} \alpha \cap \text{dom} \theta\lambda \neq \emptyset$. But $\text{rank} \alpha = 1$, so $\text{ran} \alpha \subseteq \text{dom} \theta\lambda$. Consequently, $\text{ran} \alpha = \text{ran} \alpha \cap \text{dom} \theta\lambda$, and hence $\text{dom} \alpha = (\text{ran} \alpha)\alpha^{-1} = (\text{ran} \alpha \cap \text{dom} \theta\lambda)\alpha^{-1} = \text{dom} (\alpha\theta\lambda) = \text{dom} \beta$. Now we have $\text{dom} \alpha = \text{dom} \beta$, $\text{ran} \alpha = \text{ran} \beta$ and $\text{rank} \alpha = 1$. This implies that $\beta = \alpha$.

Therefore the lemma is proved. \hfill \blacksquare

Lemma 2.3 Assume that $S(X,Y)$ covers $X$ and $Y$. If $\alpha \in S(X,Y)$ is such that $\text{ran} \alpha \cap \text{dom} \theta \neq \emptyset$, $\text{dom} \alpha \cap \text{ran} \theta \neq \emptyset$ and $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y),\theta)$, then $\text{rank} \alpha = 1$.

proof Let $x \in \text{dom} \alpha \cap \text{ran} \theta$ and $y \in \text{ran} \alpha \cap \text{dom} \theta$. Then $y'\theta = x$ and $x'\alpha = y$ for some $y' \in \text{dom} \theta$ and $x' \in \text{dom} \alpha$. This implies that $x' \in \text{dom} \alpha\theta$ and $y' \in \text{dom} \theta\alpha$. Since $S(X,Y)$ covers $X$ and $Y$, there exists a subset $A$ of $X$ such that $x'\alpha\theta \in A$ and $A_{x'} \subseteq S(X,Y)$. Consequently, $\text{ran} (\alpha\theta A_{y'}) = \{y'\}$. Since $y' \in \text{dom} \theta\alpha$, it follows that $\text{ran} (\alpha\theta A_{y'}\theta\alpha) = \{y'\theta\alpha\} = \{x\alpha\}$, so $\text{rank} (\alpha\theta A_{y'}\theta\alpha) = 1$. Thus $\alpha\theta A_{y'}\theta\alpha \neq 0$ and $\alpha\theta A_{y'}\theta\alpha \in S(X,Y)\theta\alpha \cap \alpha\theta S(X,Y) \subseteq (\alpha)_q$ in $(S(X,Y),\theta)$ by Proposition 1.1. But $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y),\theta)$, so $(\alpha\theta A_{y'}\theta\alpha)_q = (\alpha)_q$. Because $\text{rank} (\alpha\theta A_{y'}\theta\alpha) = 1$, by Lemma 2.2, we have $\alpha = \alpha\theta A_{y'}\theta\alpha$, and hence $\text{rank} \alpha = 1$. \hfill \blacksquare

Lemma 2.4 The following statements hold.

(i) $M(X,Y) \neq \emptyset$ and $M(Y,X) \neq \emptyset$ if and only if $|X| = |Y|$.

(ii) $E(X,Y) \neq \emptyset$ and $E(Y,X) \neq \emptyset$ if and only if $|X| = |Y|$.

(iii) If $\varphi$ is a bijection of $X$ onto $Y$ and $\theta \in M(Y,X)$, then $(M(X,Y),\theta) \cong (M_X,\varphi\theta)$.

(iv) If $\varphi$ is a bijection of $X$ onto $Y$ and $\theta \in E(Y,X)$, then $(E(X,Y),\theta) \cong (E_X,\varphi\theta)$.

Proof. (i) and (ii) are obvious.

(iii) and (iv). Define $\psi_1 : M(X,Y) \rightarrow M_X$ by $\alpha\psi_1 = \alpha\varphi^{-1}$ for every $\alpha \in M(X,Y)$ and define $\psi_2 : E(X,Y) \rightarrow E_X$ by $\alpha\psi_2 = \alpha\varphi^{-1}$ for every $\alpha \in E(X,Y)$. Clearly, $\psi_1$ and $\psi_2$ are isomorphisms of $(M(X,Y),\theta)$ onto $(M_X,\varphi\theta)$ where $\theta \in M(Y,X)$ and of $(E(X,Y),\theta)$ onto $(E_X,\varphi\theta)$ where $\theta \in E(Y,X)$, respectively. \hfill \blacksquare

Lemma 2.5 The following statements hold.

(i) $AM(X,Y) \neq \emptyset$ and $AM(Y,X) \neq \emptyset$ if and only if either both $X$ and $Y$ are finite or both $X$ and $Y$ are infinite and $|X| = |Y|$.

(ii) $AE(X,Y) \neq \emptyset$ and $AE(Y,X) \neq \emptyset$ if and only if either both $X$ and $Y$ are finite or both $X$ and $Y$ are infinite and $|X| = |Y|$.
**Proof.** First, let $\alpha \in AM(X,Y)$ and $\beta \in AM(Y,X)$. Then
\[ A(\alpha) = \{ x \in X \mid |(x\alpha)\alpha^{-1}| > 1 \}, \quad A(\beta) = \{ y \in Y \mid |(y\beta)\beta^{-1}| > 1 \}, \]
both $A(\alpha)$ and $A(\beta)$ are finite, $\alpha|_{X \setminus A(\alpha)} : X \setminus A(\alpha) \to Y$ is one-to-one and $\beta|_{Y \setminus A(\beta)} : Y \setminus A(\beta) \to X$ is one-to-one. Thus $|X \setminus A(\alpha)| \leq |Y|$ and $|Y \setminus A(\beta)| \leq |X|$. If $X$ is finite, then $Y$ is finite since $A(\beta)$ is finite and $|Y \setminus A(\beta)| \leq |X|$. Next, assume that $X$ is infinite. Since $A(\alpha)$ is finite, it follows that $|X| = |X \setminus A(\alpha)| \leq |Y|$, so $Y$ is infinite. But $A(\beta)$ is finite, thus $|Y| = |Y \setminus A(\beta)| \leq |X|$. Consequently, $|X| = |Y|$.

Next, let $\gamma \in AE(X,Y)$ and $\lambda \in AE(Y,X)$. Then $|Y \setminus \text{ran}\ \gamma| < \infty$ and $|X \setminus \text{ran}\ \lambda| < \infty$. If $X$ is finite, then $\text{ran}\ \gamma$ is a finite subset of $Y$, so $Y$ is finite since $|Y \setminus \text{ran}\ \gamma| < \infty$. Next, assume that $X$ is infinite. But $|X \setminus \text{ran}\ \lambda| < \infty$, thus $\text{ran}\ \lambda$ is infinite, and hence $Y$ must be infinite. Consequently,
\[ |X| = |\text{ran}\ \lambda \cup (X \setminus \text{ran}\ \lambda)| = |\text{ran}\ \lambda| + |X \setminus \text{ran}\ \lambda| \leq |Y| + |X \setminus \text{ran}\ \lambda| = |Y| = |\text{ran}\ \gamma \cup (Y \setminus \text{ran}\ \gamma)| \leq |\text{ran}\ \gamma| + |Y \setminus \text{ran}\ \gamma| \leq |X| + |Y \setminus \text{ran}\ \gamma| = |X|, \]
and thus $|X| = |Y|$.

If both $X$ and $Y$ are finite, then $AM(X,Y) = T(X,Y) = AE(X,Y)$ and $AM(Y,X) = T(Y,X) = AE(Y,X)$. If $|X| = |Y|$, then there is a bijection $\varphi$ from $X$ onto $Y$, so $\varphi \in AM(X,Y) \cap AE(X,Y)$ and $\varphi^{-1} \in AM(Y,X) \cap AE(Y,X)$.

Hence (i) and (ii) of the lemma are proved. \[\blacksquare\]

**Lemma 2.6** The following statements hold.

(i) $BL(X,Y) \neq \emptyset$ and $M(Y,X) \neq \emptyset$ if and only if $|X| = |Y|$.

(ii) $OBL(X,Y) \neq \emptyset$ and $E(Y,X) \neq \emptyset$ if and only if $|X| = |Y|$.

**Proof.** Since $BL(X,Y) \subseteq M(X,Y)$ and $OBL(X,Y) \subseteq E(X,Y)$, it follows from Lemma 2.4 that either $BL(X,Y) \neq \emptyset$ and $M(Y,X) \neq \emptyset$ or $OBL(X,Y) \neq \emptyset$ and $E(Y,X) \neq \emptyset$.

Assume that $|X| = |Y|$. By Lemma 2.4, $M(Y,X) \neq \emptyset$ and $E(Y,X) \neq \emptyset$. Since $Y$ is infinite, there are subsets $Y_1$ and $Y_2$ of $Y$ such that $Y = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \emptyset$ and $|Y_1| = |Y_2| = |Y|$. Then $|X| = |Y_1| = |Y_2|$. Let $\alpha : X \to Y_1$ be a bijection. Then $\alpha$ is one-to-one and $Y \setminus \text{ran}\ \alpha = Y \setminus Y_1 = Y_2$ which is infinite. Hence $\alpha \in BL(X,Y)$. Next, we shall show that $OBL(X,Y) \neq \emptyset$. Since $X$ is infinite, $|X \times X| = |X| = |Y|$. Then there is a bijection $\varphi : X \to X \times X$. Consequently,
\[ X = \bigcup_{x \in X} (\{x\} \times X)\varphi^{-1} \] which is a disjoint union, \[\blacksquare\]
(\{x\} \times X)\varphi^{-1} \text{ is an infinite subset of } X \text{ for every } x \in X. \quad \text{(2)}$

Let \( \psi \) be a bijection of \( X \) onto \( Y \) and define \( \beta : X \to Y \) by

\[
((\{x\} \times X)\varphi^{-1})\beta = x\psi \text{ for all } x \in X.
\]

From (1), \( \beta \) is well-defined. Since \( \text{ran } \psi = Y \), \( \text{ran } \beta = Y \) by (3). Moreover, from (1), for each \( x \in X, x \in (\{a\} \times X)\varphi^{-1} \) for some \( a \in X \), so from (3),

\[
(x\beta)\beta^{-1} = (((\{a\} \times X)\varphi^{-1})\beta)\beta^{-1} = (\{a\} \times X)\varphi^{-1}
\]

which implies by (2) that \( (x\beta)\beta^{-1} \) is infinite. Hence \( \beta \in \text{OBL}(X,Y) \).

Therefore the lemma is proved. \( \blacksquare \)

**Lemma 2.7** If \( \theta : Y \to X \) is a bijection, then \( (\text{BL}(X,Y),\theta) \cong \text{BL}_X \), through the map \( \alpha \mapsto \alpha\theta \).

**Proof.** Let \( \alpha \in \text{BL}(X,Y) \). Then \( \alpha \) is one-to-one and \( Y \setminus \text{ran } \alpha \) is infinite. Since \( \theta : Y \to X \) is a bijection, \( \alpha\theta : X \to X \) is one-to-one and

\[
X \setminus \text{ran } \alpha\theta = Y\theta \setminus (\text{ran } \alpha)\theta = (Y \setminus \text{ran } \alpha)\theta.
\]

Hence \( |X \setminus \text{ran } \alpha\theta| = |Y \setminus \text{ran } \alpha| \) since \( \theta \) is one-to-one. This shows that \( \alpha \mapsto \alpha\theta \) is a map from \( \text{BL}(X,Y) \) into \( \text{BL}_X \). This map is one-to-one since \( \theta \) is one-to-one. If \( \alpha \in \text{BL}_X \), then we can show similarly as above that \( \alpha\theta^{-1} \in \text{BL}(X,Y) \).

Also, \((\alpha\theta^{-1})\theta = \alpha \) for every \( \alpha \in \text{BL}_X \). We therefore deduce that the map \( \alpha \mapsto \alpha\theta \) is a bijection of \( \text{BL}(X,Y) \) onto \( \text{BL}_X \). This map is a homomorphism from \( (\text{BL}(X,Y),\theta) \) onto \( \text{BL}_X \) since for all \( \alpha, \beta \in \text{OBL}(X,Y) \), \((\alpha\beta)\theta = (\alpha\theta)(\beta\theta) \).

\( \blacksquare \)

**Lemma 2.8** If \( \theta : Y \to X \) is a bijection, then \( (\text{OBL}(X,Y),\theta) \cong \text{OBL}_X \), through the map \( \alpha \mapsto \alpha\theta \).

**Proof.** If \( \alpha \in \text{OBL}(X,Y) \), then \( \text{ran } \alpha\theta = X \) since \( \text{ran } \alpha = Y \) and \( \text{ran } \theta = X \), and for \( x \in X, (x\alpha\theta)(\alpha\theta)^{-1} = (x\alpha\theta)\theta^{-1}\alpha^{-1} = (x\alpha)\alpha^{-1} \) which is infinite. Hence \( \alpha \mapsto \alpha\theta \) is a map from \( \text{OBL}(X,Y) \) into \( \text{OBL}_X \). This map is one-to-one since \( \theta \) is one-to-one. If \( \alpha \in \text{OBL}_X \), then we have similarly that \( \alpha\theta^{-1} \in \text{OBL}(X,Y) \).

Also, \((\alpha\theta^{-1})\theta = \alpha \) for every \( \alpha \in \text{OBL}_X \). If \( \alpha, \beta \in \text{OBL}(X,Y) \), \((\alpha\beta)\theta = (\alpha\theta)(\beta\theta) \). Hence \( \alpha \mapsto \alpha\theta \) is an isomorphism of \( (\text{OBL}(X,Y),\theta) \) onto \( \text{OBL}_X \).

\( \blacksquare \)

**Lemma 2.9** The semigroup \( \text{BL}_X \) has no minimal quasi-ideal.

**Proof.** Let \( \alpha \in \text{BL}_X \). Then \( \alpha \) is one-to-one and \( X \setminus \text{ran } \alpha \) is infinite. Since \( \alpha^2 \in (\alpha)_q, (\alpha^2)_q \subseteq (\alpha)_q \). We claim that \( (\alpha^2)_q \not\subseteq (\alpha)_q \). Suppose on the contrary that \( (\alpha^2)_q = (\alpha)_q \). By Proposition 1.1, \( \alpha = \alpha^2 \) or \( \alpha = \beta\alpha^2 \) for some \( \beta \in \text{BL}_X \). But \( \alpha \) is one-to-one, so we have \( 1_X = \alpha \) or \( 1_X = \beta\alpha \) which implies that \( \text{ran } \alpha = X \). This is contrary to the fact that \( X \setminus \text{ran } \alpha \) is infinite. Hence we have the claim. Since \( \alpha \in \text{BL}_X \) is arbitrary, we deduce that \( \text{BL}_X \) has no minimal quasi-ideal. \( \blacksquare \)
Lemma 2.10 The semigroup $OBL_X$ has no minimal quasi-ideal.

Proof. Let $\alpha \in OBL_X$. Then $\text{ran} \alpha = X$ and $|(x\alpha)\alpha^{-1}| > 1$ for all $x \in X$ and $(\alpha^2)_q \subseteq (\alpha)_q$. Suppose that $(\alpha^2)_q = (\alpha)_q$. By Proposition 1.1, $\alpha = \alpha^2$ or $\alpha = \alpha^2\beta$ for some $\beta \in OBL_X$. Since $\alpha$ is onto, $1_X = \alpha$ or $1_X = \alpha\beta$. Thus $\alpha$ is one-to-one. This is contrary to the fact that $|(x\alpha)\alpha^{-1}| > 1$ for all $x \in X$. Hence $(\alpha^2)_q \subsetneq (\alpha)_q$. Therefore $OBL_X$ has no minimal quasi-ideal.

3 Generalized Semigroups of $P_X, T_X$ and $I_X$

In this section, we characterize minimal quasi-ideals of the semigroup $(S(X,Y), \theta)$ where $S(X,Y)$ is $P(X,Y), T(X,Y)$ or $I(X,Y)$ and $\theta \in S(Y,X)$. It is clearly seen that all $P(X,Y), T(X,Y)$ and $I(X,Y)$ cover $X$ and $Y$.

Theorem 3.1 Let $S(X,Y)$ be $P(X,Y)$ or $I(X,Y)$ and $\theta \in S(Y,X)$. Then for $\alpha \in S(X,Y) \setminus \{0\}$, $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y), \theta)$ if and only if one of the following statements holds.

(i) $\text{ran} \alpha \cap \text{dom} \theta = \emptyset$.

(ii) $\text{dom} \alpha \cap \text{ran} \theta = \emptyset$.

(iii) $\text{rank} \alpha = 1$.

If this is the case, $(\alpha)_q = \{0, \alpha\}$. If $\alpha \theta \alpha = 0$, then $(\alpha)_q$ is a zero subsemigroup of $(S(X,Y), \theta)$. Otherwise, $(\alpha)_q$ is a subgroup with zero of $(S(X,Y), \theta)$.

Proof. To show sufficiency, first assume that (i) or (ii) holds. Then by Lemma 2.1, $(\alpha)_q = \{0, \alpha\}$ in $(S(X,Y), \theta)$, so $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y), \theta)$. If $\text{rank} \alpha = 1$, by Lemma 2.2, $(\alpha)_q = \{0, \alpha\}$ in $(S(X,Y), \theta)$, so it is a minimal quasi-ideal of $(S(X,Y), \theta)$. Clearly, if $\alpha \theta \alpha = 0$, then $(\alpha)_q$ is a zero subsemigroup of $(S(X,Y), \theta)$. By Theorem 1.3, $(\alpha)_q$ is a subgroup with zero of $(S(X,Y), \theta)$ if $\alpha \theta \alpha \neq 0$.

To show necessity, assume that $(\alpha)_q$ is a minimal quasi-ideal of $(S(X,Y), \theta)$ and suppose that (i) and (ii) are false. Then $\text{ran} \alpha \cap \text{dom} \theta \neq \emptyset$ and $\text{dom} \alpha \cap \text{ran} \theta \neq \emptyset$. But $S(X,Y)$ covers $X$ and $Y$, so by Lemma 2.3, we deduce that $\text{rank} \alpha = 1$. Hence (iii) holds.

Therefore the theorem is proved.

Theorem 3.2 Let $\theta \in T(Y,X)$. Then for $\alpha \in T(X,Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(T(X,Y), \theta)$ if and only if $\text{rank} \alpha = 1$.

If this is the case, $(\alpha)_q = \{\alpha\}$.

Proof. Since $\text{dom} \theta = Y \supseteq \text{ran} \beta$ and $\text{dom} \beta = X \supseteq \text{ran} \theta \neq \emptyset$ for every $\beta \in T(X,Y)$, we have from Lemma 2.3 that if $(\alpha)_q$ is a minimal quasi-ideal of $(T(X,Y), \theta)$, then $\text{rank} \alpha = 1$.

For the converse, assume that $\text{rank} \alpha = 1$. Then $(\alpha)_q = \{\alpha\}$ by Lemma 2.2, so $(\alpha)_q$ is a minimal quasi-ideal of $(T(X,Y), \theta)$.
4 Generalized Semigroups of \( M_X, E_X, AM_X \) and \( AE_X \)

We show that the semigroup \((S(X,Y),\theta)\) where \(S(X,Y)\) is any of \(M(X,Y)\), \(E(X,Y)\), \(AM(X,Y)\) or \(AE(X,Y)\) and \(\theta \in S(Y,X)\) has no minimal quasi-ideals except for the case that \(X\) and \(Y\) are finite.

**Theorem 4.1** The semigroup \((M(X,Y),\theta)\), where \(\theta \in M(Y,X)\), has a minimal quasi-ideal if and only if \(|X| = |Y| < \infty\).

If \(|X| = |Y| < \infty\), then \((M(X,Y),\theta)\) is a group, so \(M(X,Y)\) is itself a unique minimal quasi-ideal of \((M(X,Y),\theta)\).

**Proof.** By Lemma 2.4(i), \(|X| = |Y|\), and from Lemma 2.4(iii), \((M(X,Y),\theta) \cong (M_X, \varphi \theta)\) where \(\varphi : X \rightarrow Y\) is a bijection. If \(|X| < \infty\), then \(M_X = G_X\), so \((M_X, \varphi \theta)\) is a group and hence \(M_X\) is a unique minimal quasi-ideal of \((M_X, \varphi \theta)\).

Then to prove the theorem, it remains to show that \((M_X, \varphi \theta)\) has no minimal quasi-ideals if \(X\) is infinite. Let \(\overline{\theta} = \varphi \theta\) and assume that \(X\) is infinite. Let \(a \in X\). Then \(|X| = |X \setminus \{a\}|\) since \(X\) is infinite. Then there is a one-to-one map \(\beta\) from \(X\) onto \(X \setminus \{a\}\), so \(\beta \in M_X\). Let \(\alpha \in M_X\). From Proposition 1.1, we have that \(a \beta \overline{\alpha} \in (\alpha)_q\) in \((M_X, \overline{\theta})\), so \((a \beta \overline{\alpha})_q \subseteq (\alpha)_q\). If \(\alpha \in (a \beta \overline{\alpha})_q\), then from Proposition 1.1, \(\alpha = a \beta \overline{\alpha}\) or \(\alpha = \gamma a \beta \overline{\alpha}\) for some \(\gamma \in M_X\). Since \(\alpha\) is one-to-one, we have \(1_X = a \beta \overline{\alpha}\) or \(1_X = \gamma a \beta \overline{\alpha}\) where \(1_X\) is the identity map on \(X\). This implies that \(\text{ran} (\beta \overline{\theta}) = X\). Hence

\[ X = X \beta \overline{\theta} = (X \beta) \overline{\theta} = (X \setminus \{a\}) \overline{\theta}, \]

so \(a \overline{\theta} = b \overline{\theta}\) for some \(b \in X \setminus \{a\}\) which is contrary to that \(\overline{\theta} : X \rightarrow X\) is one-to-one. This proves that \((a \beta \overline{\alpha})_q \subsetneq (\alpha)_q\). Hence we deduce that for every \(\alpha \in M_X\), \((\alpha)_q\) is not a minimal quasi-ideal of \((M_X, \overline{\theta})\). Therefore \((M_X, \overline{\theta})\) has no minimal quasi-ideals.

Therefore the theorem is proved. \(\blacksquare\)

**Theorem 4.2** The semigroup \((E(X,Y),\theta)\), where \(\theta \in E(Y,X)\), has a minimal quasi-ideal if and only if \(|X| = |Y| < \infty\).

If \(|X| = |Y| < \infty\), then \((E(X,Y),\theta)\) is a group, so \(E(X,Y)\) is itself a unique minimal quasi-ideal of \((E(X,Y),\theta)\).

**Proof.** It follows from Lemma 2.4(ii) and (iv) that \(|X| = |Y|\) and \((E(X,Y),\theta) \cong (E_X, \varphi \theta)\) where \(\varphi : X \rightarrow Y\) is a bijection. If \(|X| < \infty\), then \((E_X, \varphi \theta) = (G_X, \varphi \theta)\) which is a group, and thus \(E_X\) is a unique minimal quasi-ideal of \((E_X, \varphi \theta)\). Let \(\overline{\theta} = \varphi \theta\). Then to prove the theorem it remains to prove that if \(X\) is infinite, then \((E_X, \overline{\theta})\) has no minimal quasi-ideals. Assume that \(X\) is
infinite. Let $a$, $b$ and $c$ be distinct elements of $X$. Then $|X \setminus \{a, b\}| = |X \setminus \{c\}|$. Let $\psi$ be a bijection of $X \setminus \{a, b\}$ onto $X \setminus \{c\}$. Let $\beta: X \to X$ be defined by

$$a\beta = b\beta = c$$

and $x\beta = x\psi$ for all $x \in X \setminus \{a, b\}$.

Then $\beta \in E_X$ but $\beta$ is not one-to-one. If $\alpha \in E_X$, then from Proposition 1.1, we have $\alpha \beta \bar{\beta} \alpha \in (\alpha)_q$ in $(E_X, \bar{\beta})$, and thus $(\alpha \beta \bar{\beta} \alpha)_q \subseteq (\alpha)_q$, suppose that $\alpha \in (\alpha \beta \bar{\beta} \alpha)_q$. By Proposition 1.1, $\alpha = \alpha \beta \bar{\beta} \alpha$ or $\alpha = \alpha \bar{\beta} \alpha \beta \bar{\gamma}$ for some $\gamma \in E_X$. Since ran $\alpha = X$, it follows that $1_X = \bar{\beta} \bar{\alpha}$ or $1_X = \beta \alpha \bar{\alpha} \gamma$. This implies that $\bar{\gamma}$ must be one-to-one, so $\bar{\gamma} \in G_X$. Hence $\beta \bar{\alpha} = (\bar{\gamma})^{-1}$ or $\beta \bar{\alpha} \bar{\gamma} = (\bar{\gamma})^{-1}$ which implies that $\beta$ must be one-to-one, a contradiction. This proves that $(\alpha \beta \bar{\beta} \alpha)_q \not\subseteq (\alpha)_q$. Therefore we deduce that for every $\alpha \in E_X$, $(\alpha)_q$ is not a minimal quasi-ideal of $(E_X, \bar{\beta})$. This shows that $(E_X, \bar{\beta})$ has no minimal quasi-ideals.

Hence the theorem is completely proved.

Theorem 4.3 For $\theta \in AM(Y, X)$, the semigroup $(AM(X, Y), \theta)$ has a minimal quasi-ideal if and only if $X$ and $Y$ are finite.

If $X$ and $Y$ are finite, then for $\alpha \in AM(X, Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(AM(X, Y), \theta)$ if and only if rank $\alpha = 1$. If this is the case, $(\alpha)_q = \{\alpha\}$.

Proof. First assume that $X$ and $Y$ are finite. Then $AM(X, Y) = T(X, Y)$ and $AM(Y, X) = T(Y, X)$. By Theorem 3.2, for every $\alpha \in AM(X, Y)$, $(\alpha)_q$ is a minimal quasi-ideal of $(AM(X, Y), \theta)$ if and only if rank $\alpha = 1$, and for this case, $(\alpha)_q = \{\alpha\}$.

To prove the converse, assume that $X$ or $Y$ is not finite. By Lemma 2.5(i), $X$ and $Y$ are infinite and $|X| = |Y|$. Let $\alpha \in AM(X, Y)$. Then $\alpha \theta \in AM_X$ which implies that $\alpha \theta: X \setminus A(\alpha \theta) \to X$ is one-to-one. Let $a$ and $b$ be distinct elements of $X \setminus A(\alpha \theta)$. Then $a a \theta \neq b a \theta$, so $a a \neq b a$. Let $c \in Y$. Since $|X \setminus \{a a \theta, b a \theta\}| = |X| = |Y| = |Y \setminus \{c\}|$, there is a bijection $\varphi$ from $X \setminus \{a a \theta, b a \theta\}$ onto $Y \setminus \{c\}$ and define $\beta: X \to Y$ by

$$x\beta = \begin{cases} c & \text{if } x = a a \theta \text{ or } x = b a \theta, \\ x \varphi & \text{if } x \in X \setminus \{a a \theta, b a \theta\}. \end{cases}$$

Then $A(\beta) = \{a a \theta, b a \theta\}$, so $\beta \in AM(X, Y)$. Now we have $a a \theta \beta \theta a = b a \theta \beta \theta a$ and $a a \neq b a$, so $\alpha \neq a a \theta \beta \theta a$. By Proposition 1.1, $\alpha \beta \beta \alpha \in (\alpha)_q$. Hence $(\alpha \beta \beta \alpha)_q \subseteq (\alpha)_q$. Suppose that $\alpha \in (\alpha \beta \beta \alpha)_q$. But $\alpha \neq a a \theta \beta \theta a$, so from Proposition 1.1, $\alpha = a a \theta \beta \theta \alpha \gamma$ for some $\gamma \in AM(X, Y)$. But $a a \theta \beta \theta \alpha = b a \theta \beta \theta a$, so $a a = (a a \theta \beta \theta a) \theta \gamma = (b a \theta \beta \theta a) \theta \gamma = b a$. This is a contradiction. Hence $(\alpha \beta \beta \alpha)_q \not\subseteq (\alpha)_q$ which implies that $(\alpha)_q$ is not a minimal quasi-ideal of $(AM(X, Y), \theta)$.

Therefore the proof is complete. ■
Theorem 4.4 For \( \theta \in AE(Y, X) \), the semigroup \((AE(X, Y), \theta)\) has a minimal quasi-ideal if and only if \( X \) and \( Y \) are finite.

If \( X \) and \( Y \) are finite, then for \( \alpha \in AE(X, Y) \), \((\alpha)_{q} \) is a minimal quasi-ideal of \((AE(X, Y), \theta)\) if and only if \( \text{rank} \alpha = 1 \). If this is the case, \((\alpha)_{q} = \{ \alpha \} \).

**Proof.** If \( X \) and \( Y \) are finite, then \( AE(X, Y) = T(X, Y) \) and \( AE(Y, X) = T(Y, X) \), so by Theorem 3.2, for \( \alpha \in AE(X, Y) \), \((\alpha)_{q} \) is a minimal quasi-ideal of \((AE(X, Y), \theta)\) if and only if \( \text{rank} \alpha = 1 \) and for this case, \((\alpha)_{q} = \{ \alpha \} \).

Conversely, assume that \( X \) or \( Y \) is not finite. By Lemma 2.5(ii), \( X \) and \( Y \) are infinite and \( |X| = |Y| \). Let \( \alpha \in AE(X, Y) \). Then \( \alpha \theta \in AE_{X} \), so \( X \setminus \text{ran} \alpha \theta \) is finite. Hence \( \text{ran} \alpha \theta \) is infinite. Let \( a, b \in X \) be such that \( a\alpha \neq b\alpha \).

Then \( a\alpha \neq b\alpha \) and \( |X \setminus \{a\alpha, b\alpha\}| = |Y| \). Let \( \varphi : X \setminus \{a\alpha, b\alpha\} \to Y \) be a bijection and define \( \beta : X \to Y \) by

\[
x\beta = \begin{cases} 
a\alpha & \text{if } x = a\alpha \theta \text{ or } x = b\alpha \theta, 
\varphi x & \text{if } x \in X \setminus \{a\alpha, b\alpha\}.
\end{cases}
\]

Then \( \beta \in E(X, Y) \subseteq AE(X, Y) \) and \( a\alpha \theta \beta \alpha = b\alpha \beta \theta \alpha \). But \( a\alpha \neq b\alpha \), so \( \alpha \neq a\theta \beta \alpha \). By Proposition 1.1, \( a\theta \beta \alpha \in (\alpha)_{q} \), so \( (a\theta \beta \alpha)_{q} \subseteq (\alpha)_{q} \). If \( \alpha \in (a\theta \beta \alpha)_{q} \), then by Proposition 1.1, \( \alpha = a\theta \beta \alpha \theta \gamma \) for some \( \gamma \in AE(X, Y) \) since \( \alpha \neq a\theta \beta \alpha \). This implies that \( a\alpha = (a\alpha \theta \beta \alpha)\theta \gamma = (b\alpha \beta \theta \alpha)\theta \gamma = b\alpha \), a contradiction. Consequently, \( (a\theta \beta \alpha)_{q} \not\subseteq (\alpha)_{q} \), so \((\alpha)_{q} \) is not a minimal quasi-ideal of \((AE(X, Y), \theta)\).

Hence the theorem is proved. \( \square \)

5 Generalized Semigroups of \( BL_{X} \) and \( OBL_{X} \)

In this section, we show that the semigroup \((BL(X, Y), \theta)\) where \( \theta \in M(Y, X) \) and the semigroup \((OBL(X, Y), \theta)\) where \( \theta \in E(Y, X) \) have no minimal quasi-ideals.

**Theorem 5.1** For \( \theta \in M(Y, X) \), the semigroup \((BL(X, Y), \theta)\) has no minimal quasi-ideal.

**Proof.** Let \( \theta \in M(Y, X) \). Then \( \theta : Y \to X \) is one-to-one.

**Case 1:** \( \theta \) is onto. Then \( \theta : Y \to X \) is a bijection. By Lemma 2.7, \((BL(X, Y), \theta) \cong BL_{X} \). But \( BL_{X} \) has no minimal quasi-ideal by Lemma 2.9, so \((BL(X, Y), \theta) \) has no minimal quasi-ideal.

**Case 2:** \( \theta \) is not onto. Let \( \alpha \in BL(X, Y) \). Then \( \alpha \theta \alpha \in (\alpha)_{q} \), and so \((\alpha \theta \alpha)_{q} \subseteq (\alpha)_{q} \). Suppose that \((\alpha \theta \alpha)_{q} = (\alpha)_{q} \). By Proposition 1.1, \( \alpha = a\theta \alpha \) or \( \alpha = b\theta \alpha \theta \alpha \) for some \( \beta \in BL(X, Y) \). Since \( \alpha \) is one-to-one, \( 1_{X} = a\theta \) or \( 1_{X} = b\theta \alpha \theta \) which implies that \( \theta \) is onto, a contradiction. This shows that
(αθα)_q \subsetneq (α)_q. We then deduce that \((BL(X,Y),\theta)\) has no minimal quasi-ideal.

Hence the theorem is proved, as desired. ■

**Theorem 5.2** For \(\theta \in E(Y,X)\), the semigroup \(OBL(X,Y),\theta)\) has no minimal quasi-ideal.

**Proof.** Let \(\theta \in E(Y,X)\). Then \(\theta : Y \to X\) is onto.

**Case 1:** \(\theta\) is one-to-one. Then \(\theta : Y \to X\) is a bijection. By Lemma 2.8, \((OBL(X,Y),\theta) \cong OBL_X\). But from Lemma 2.10, \(OBL_X\) has no minimal quasi-ideal, so \((OBL(X,Y),\theta)\) has no minimal quasi-ideal.

**Case 2:** \(\theta\) is not one-to-one. Let \(\alpha \in OBL(X,Y)\). Then \(ran \alpha = Y\) and \((x\alpha)\alpha^{-1}\) is infinite for every \(x \in X\). Since \(\alpha\theta\alpha \in (\alpha)_q\), \((\alpha\theta\alpha)_q \subseteq (\alpha)_q\). \((\alpha\theta\alpha)_q = (\alpha)_q\). By Proposition 1.1, \(\alpha = \alpha\theta\alpha\) or \(\alpha = \alpha\theta\alpha\beta\) for some \(\beta \in OBL(X,Y)\). Since \(\alpha\) is onto, \(1_Y = \theta\alpha\) or \(1_X = \theta\alpha\beta\) which implies that \(\theta\) is one-to-one, a contradiction. This shows that \((\alpha\theta\alpha)_q \subsetneq (\alpha)_q\). We then deduce that \((OBL(X,Y),\theta)\) has no minimal quasi-ideal.

Hence the theorem is proved, as desired. ■

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**References**


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