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An Equation Related to θ-Centralizers in Semiprime Rings

M. N. Daif

Department of Mathematics, Faculty of Science
Al-Azhar University, Nasr City (11884), Cairo, Egypt
nagydaif@yahoo.com

M. S. Tammam El-Sayiad

Department of Mathematics, Carver Hall
Iowa State University of Science and Technology
Ames 50011-2064, IA, USA
mohammad@iastate.edu

N. M. Muthana

Department of Mathematics, Faculty of Education
King Abdul Aziz University, Jeddah, Saudi Arabia
najat_muthana@hotmail.com

Abstract

The purpose of this note is to prove the following result. Let R be a 2-torsion free semiprime ring and let $T : R \longrightarrow R$ be an additive mapping. Suppose that $3T(xyx) = T(x)\theta(yx) + \theta(x)T(y)\theta(x) + \theta(xy)T(x)$ and $\theta(x)T(xy + yx)\theta(x) = \theta(x)T(y)\theta(x^2) + \theta(x^2)T(y)\theta(x)$ hold for all $x, y \in R$, where θ is a homomorphism from R onto R. Then T is a $\theta-$centralizer of R.

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1 Introduction

This note is motivated by the work of Vukman and Kosi-Ulbl ([8]). Throughout this note, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where n is an integer, in case $nx = 0$, $x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use the basic commutator identities $[x, yz] = [x, y]z + y[x, z]$ and $[xz, y] = [x, y]z + x[z, y]$.

Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \longrightarrow R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$, and called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. A derivation D is inner in case there exists an $a \in R$ such that $D(x) = [a, x]$ for all $x \in R$. Every derivation is a Jordan derivation. The inverse is in general not true. A classical result of Herstein ([7]) asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in ([3]). Cusack ([6]) has generalized Herstein's result to 2-torsion free semiprime rings (see also [4] for an alternative proof).

An additive mapping $T : R \longrightarrow R$ is called a left (right) centralizer in case $T(xy) = T(x)y \quad (T(xy) = xT(y))$ holds for all $x, y \in R$. We follow Zalar ([12]) and call T a centralizer in case T is both a left and a right centralizer. If $a \in R$ then $L_a(x) = ax$ is a left centralizer and $R_a(x) = xa$ is a right centralizer. An additive mapping $T : R \longrightarrow R$ is called a left (right) Jordan centralizer in case $T(x^2) = T(x)x \quad (T(x^2) = xT(x))$.

Following ideas from ([4]), Zalar ([12]) has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Also, Vukman ([10]) proved that if $T : R \longrightarrow R$ is an additive mapping such that $2T(x^2) = T(x)x + xT(x)$ holds for all $x \in R$, then T is a centralizer. Vukman ([11]), also proved that if R is a 2-torsion free semiprime ring and $T : R \longrightarrow R$ is an additive mapping such that $T(xyx) = xT(y)x$ holds for all $x, y \in R$, then T is a centralizer.

In ([9]), Vukman and Irena proved that if R is a 2-torsion free semiprime ring and $T : R \longrightarrow R$ is an additive mapping such that $2T(xyx) = T(x)yx + xyT(x)$ holds for all $x, y \in R$, then T is a centralizer. In ([8]) they proved that if R is a 2-torsion free semiprime ring and $T : R \longrightarrow R$ is an additive mapping such that $3T(xyx) = T(x)yx + xT(y)x + xyT(x)$ holds for all $x, y \in R$, then T is a centralizer and $T(x) = \lambda x$ where λ is in the extended centroid of R. We have not been able to verify the linearization on Page 259 line -6 which gives equation [50] in ([8]).

An additive mapping $D : R \longrightarrow R$, where R is an arbitrary ring, is a Jordan triple derivation in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all $x, y \in R$. One can easily prove that any Jordan triple derivation is a triple derivation (see[3]). Bresar ([5]) has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a triple derivation.

In ([1]), Albas has introduced the notation of θ-centralizer and Jordan θ-centralizer, which is a generalization of centralizer and Jordan centralizer. He proved, on a 2-torsion free semiprime ring with some conditions, that every Jordan θ-centralizer is a θ-centralizer. In this note, Vukman and Irena's identity ([8]) will be studied in the sense of Albas's definition.

Definition 1.1. *[1] An additive mapping $T : R \longrightarrow R$ is called a left (right) $\theta-$centralizer associated with a homomorphism $\theta : R \longrightarrow R$ if for all $x, y \in R$,*

$$T(xy) = T(x)\theta(y) \quad (T(xy) = \theta(x)T(y)).$$

T is called a left (right) Jordan $\theta-$centralizer if for all $x \in R$,

$$T(x^2) = T(x)\theta(x) \quad (T(x^2) = \theta(x)T(x)).$$

Remark. Every centralizer is a special case of a θ-centralizer with $\theta = I_R$.

If $T : R \longrightarrow R$ is a θ-centralizer associated with a homomorphism $\theta : R \longrightarrow R$, where R is an arbitrary ring, then T satisfies the relations

$$3T(xyx) = T(x)\theta(yx) + \theta(x)T(y)\theta(x) + \theta(xy)T(x) \quad \forall\, x, y \in R. \tag{1}$$

And,

$$\theta(x)T(xy + yx)\theta(x) = \theta(x)T(y)\theta(x^2) + \theta(x^2)T(y)\theta(x) \quad \forall\, x, y \in R. \tag{2}$$

It seems natural to ask whether the converse is true. More precisely, we are asking whether an additive mapping T on a ring R satisfying relations (1) and (2) is a θ-centralizer. It is the aim in this paper to prove that the answer is affirmative when R is a 2-torsion free semiprime ring and θ is a surjective homomorphism.

2 The Main Result

We now give the main result of this paper.

Theorem 2.1. *Let R be a 2-torsion free semiprime ring and let $T : R \longrightarrow R$ be an additive mapping. suppose that $3T(xyx) = T(x)\theta(yx) + \theta(x)T(y)\theta(x) + \theta(xy)T(x)$ and $\theta(x)T(xy + yx)\theta(x) = \theta(x)T(y)\theta(x^2) + \theta(x^2)T(y)\theta(x)$ hold for all pairs $x, y \in R$ and $\theta(Z(R)) = Z(R)$, where θ be a nonzero surjective endomorphism on R. Then T is a θ-centralizer.*

For the proof of theorem (2.1) the following Lemma will be needed which can be found in ([11, Lemma 1]).

Lemma 2.2. *Let R be a semiprime ring. Suppose that the relation $axb + bxc = 0$ holds for all $x \in R$ and some $a, b, c \in R$. In this case $(a + c)xb = 0$ is satisfied for all $x \in R$.*

Proof. of Theorem (2.1): After replacing x by $x + z$ in (1), we obtain

$$\begin{aligned} 3T(xyz + zyx) &= T(x)\theta(yz) + T(z)\theta(yx) + \theta(x)T(y)\theta(z) \\ &+ \theta(z)T(y)\theta(x) + \theta(zy)T(x) + \theta(xy)T(z), \\ &\quad \forall x, y, z \in R, \end{aligned} \tag{3}$$

For $z = x^2$ we have

$$\begin{aligned} 3T(xyx^2 + x^2yx) &= T(x)\theta(yx^2) + T(x^2)\theta(yx) + \theta(x)T(y)\theta(x^2) \\ &+ \theta(x^2)T(y)\theta(x) + \theta(x^2y)T(x) + \theta(xy)T(x^2), \\ &\quad \forall x, y \in R. \end{aligned} \tag{4}$$

After replacing y by $xy + yx$ in (1) we obtain

$$\begin{aligned} 3T(xyx^2 + x^2yx) &= T(x)\theta(xyx) + T(x)\theta(yx^2) \\ &+ \theta(x)T(xy + yx)\theta(x) + \theta(x^2y)T(x) \\ &+ \theta(xyx)T(x), \qquad \forall x, y \in R. \end{aligned} \tag{5}$$

After subtracting (5) from (4), we arrive at

$$\begin{aligned} A(x)\theta(yx) &+ \theta(xy)B(x) \\ +\theta(x)(T(y)\theta(x) + \theta(x)T(y) &- T(xy + yx))\theta(x) = 0, \qquad \forall x, y \in R, \end{aligned} \tag{6}$$

where $A(x) = T(x^2) - T(x)\theta(x)$, and $B(x) = T(x^2) - \theta(x)T(x)$.
Letting $y = x$ and $z = y$ in (3) gives

$$\begin{aligned} 3T(x^2y + yx^2) &= T(x)\theta(xy) + T(y)\theta(x^2) \\ &+ \theta(x)T(x)\theta(y) + \theta(y)T(x)\theta(x) + \theta(yx)T(x) \\ &+ \theta(x^2)T(y), \qquad \forall x, y \in R. \end{aligned} \tag{7}$$

After replacing x by $3x$ and z by x^3 in (3) and using (1) we obtain

$$\begin{aligned} 9T(xyx^3 + x^3yx) &= 3T(x)\theta(yx^3) + 3T(x^3)\theta(yx) \\ &+ 3\theta(x)T(y)\theta(x^3) + 3\theta(x^3)T(y)\theta(x) \\ &+ 3\theta(x^3y)T(x) + 3\theta(xy)T(x^3) \\ &= 3T(x)\theta(yx^3) + T(x)\theta(x^2yx) + \theta(x)T(x)\theta(xyx) \\ &+ \theta(x^2)T(x)\theta(yx) + 3\theta(x)T(y)\theta(x^3) \\ &+ 3\theta(x^3)T(y)\theta(x) + \theta(xy)T(x)\theta(x^2) \\ &+ \theta(xyx)T(x)\theta(x) + \theta(xyx^2)T(x) \\ &+ 3\theta(x^3y)T(x), \qquad \forall x, y \in R. \end{aligned} \tag{8}$$

Replacing y by $3(x^2y+yx^2)$ in (1) and using (7) we obtain

$$\begin{aligned}
9T(xyx^3+x^3yx) &= 3T(x)\theta(x^2y+yx^2)\theta(x)+3\theta(x)T(x^2y+yx^2)\theta(x)\\
&+ 3\theta(x)\theta(x^2y+yx^2)T(x)\\
&= 3T(x)\theta(x^2y+yx^2)\theta(x)+\theta(x)(T(x)\theta(xy)+T(y)\theta(x^2)\\
&+ \theta(x)T(x)\theta(y)+\theta(y)T(x)\theta(x)+\theta(x^2)T(y)\\
&+ \theta(yx)T(x))\theta(x)+3\theta(x)\theta(x^2y+yx^2)T(x)\\
&= 3T(x)\theta(x^2yx)+3T(x)\theta(yx^3)+\theta(x)T(x)\theta(xyx)\\
&+ \theta(x)T(y)\theta(x^3)+\theta(x^2)T(x)\theta(yx)+\theta(xy)T(x)\theta(x^2)\\
&+ \theta(x^3)T(y)\theta(x)+\theta(xyx)T(x)\theta(x)+3\theta(x^3y)T(x)\\
&+ 3\theta(xyx^2)T(x), \qquad \forall x,y\in R.
\end{aligned}\tag{9}$$

Subtracting (9) from (8) we obtain

$$\begin{aligned}
T(x)\theta(x^2yx) &+ \theta(xyx^2)T(x)-\theta(x^3)T(y)\theta(x)\\
&- \theta(x)T(y)\theta(x^3)=0, \qquad \forall x,y\in R.
\end{aligned}\tag{10}$$

Replacing y by $3xyx$ in (7) we obtain

$$\begin{aligned}
9T(x^3yx+xyx^3) &= 3T(x)\theta(x^2yx)+T(x)\theta(yx^3)\theta(x)T(y)\theta(x^3)\\
&+ \theta(xy)T(x)\theta(x^2)+3\theta(x)T(x)\theta(xyx)\\
&+ 3\theta(xyx)T(x)\theta(x)+\theta(x^2)T(x)\theta(yx)\\
&+ \theta(x^3)T(y)\theta(x)+\theta(x^3y)T(x)\\
&+ 3\theta(xyx^2)T(x), \qquad \forall x,y\in R.
\end{aligned}\tag{11}$$

On the other hand by replacing z by $3x^3$ in (3) we obtain;

$$\begin{aligned}
9T(x^3yx+xyx^3) &= 3T(x)\theta(yx^3)+T(x)\theta(x^2yx)\\
&+ \theta(x)T(x)\theta(xyx)+\theta(x^2)T(x)\theta(yx)\\
&+ 3\theta(x)T(y)\theta(x^3)+3\theta(x^3)T(y)\theta(x)\\
&+ \theta(xy)T(x)\theta(x^2)+\theta(xyx)T(x)\theta(x)\\
&+ \theta(xyx^2)T(x)+3\theta(x^3y)T(x), \qquad \forall x,y\in R.
\end{aligned}\tag{12}$$

Comparing (11) and (12) we arrive at

$$\begin{aligned}
&T(x)\theta(yx^3)-T(x)\theta(x^2yx)+\theta(x)T(y)\theta(x^3)\\
-\ &\theta(x)T(x)\theta(xyx)-\theta(xyx^2)T(x)+\theta(x^3y)T(x)\\
-\ &\theta(xyx)T(x)\theta(x)+\theta(x^3)T(y)\theta(x)=0, \qquad \forall x,y\in R.
\end{aligned}\tag{13}$$

From (10) and (13) we obtain

$$\begin{aligned}
&T(x)\theta(yx^3)-\theta(x)T(x)\theta(xyx)\\
+\ &\theta(x^3y)T(x)-\theta(xyx)T(x)\theta(x)=0, \qquad \forall x,y\in R.
\end{aligned}\tag{14}$$

Replacing y by yx in the above relation gives

$$\begin{aligned} & T(x)\theta(yx^4) - \theta(x)T(x)\theta(xyx^2) \\ + \quad & \theta(x^3yx)T(x) - \theta(xyx^2)T(x)\theta(x) = 0, \qquad \forall x, y \in R. \end{aligned} \tag{15}$$

On the other hand right multiplication of (14) by $\theta(x)$ gives

$$\begin{aligned} T(x)\theta(yx^4) \quad - \quad & \theta(x)T(x)\theta(xyx^2) + \theta(x^3y)T(x)\theta(x) \\ - \quad & \theta(xyx)T(x)\theta(x^2) = 0, \qquad \forall x, y \in R. \end{aligned} \tag{16}$$

Subtracting (16) from (15) gives

$$\theta(x^3y)[T(x), \theta(x)] - \theta(xyx)[T(x), \theta(x)]\theta(x) = 0, \qquad \forall x, y \in R. \tag{17}$$

Left multiplication of (17) by $T(x)$ gives

$$\begin{aligned} T(x)\theta(x^3y)[T(x), \theta(x)] \quad - \quad & T(x)\theta(xyx)[T(x), \theta(x)]\theta(x) = 0, \\ & \forall x, y \in R. \end{aligned} \tag{18}$$

Replacing $\theta(y)$ by $T(x)\theta(y)$ in (17) gives,

$$\begin{aligned} \theta(x^3)T(x)\theta(y)[T(x), \theta(x)] \quad - \quad & \theta(x)T(x)\theta(yx)[T(x), \theta(x)]\theta(x) = 0, \\ & \forall x, y \in R. \end{aligned} \tag{19}$$

After subtracting (19) from (18) we arrive at

$$\begin{aligned} [T(x), \theta(x^3)]\theta(y)[T(x), \theta(x)] \quad - \quad & [T(x), \theta(x)]\theta(yx)[T(x), \theta(x)]\theta(x) = 0, \\ & \forall x, y \in R. \end{aligned} \tag{20}$$

In the above relation let

$$a = [T(x), \theta(x^3)], b = [T(x), \theta(x)], c = -\theta(x)[T(x), \theta(x)]\theta(x) \text{ and } z = \theta(y)$$

From the above substitutions we have

$$azb + bzc = 0.$$

We apply Lemma (2.2) to the above relation to obtain

$$\{[T(x), \theta(x^3)] - \theta(x)[T(x), \theta(x)]\theta(x)\}\theta(y)[T(x), \theta(x)] = 0, \qquad \forall x, y \in R,$$

this reduces to

$$\begin{aligned} \{[T(x), \theta(x)]\theta(x^2) \quad + \quad & \theta(x^2)[T(x), \theta(x)]\}\theta(y)[T(x), \theta(x)] = 0, \\ & \forall x, y \in R. \end{aligned} \tag{21}$$

Right multiplication of the above relation by $\theta(x^2)$ gives

$$\begin{aligned}\{[T(x),\theta(x)]\theta(x^2) \ &+\ \theta(x^2)[T(x),\theta(x)]\}\theta(y)[T(x),\theta(x)]\theta(x^2)=0,\\ &\forall x,y\in R.\end{aligned} \tag{22}$$

After replacing $\theta(y)$ by $\theta(y)\theta(x^2)$ in (21) we get,

$$\begin{aligned}\{[T(x),\theta(x)]\theta(x^2) \ &+\ \theta(x^2)[T(x),\theta(x)]\}\theta(y)\theta(x^2)[T(x),\theta(x)]=0,\\ &\forall x,y\in R.\end{aligned} \tag{23}$$

Adding (22) to (23) we obtain

$$\begin{gathered}\{[T(x),\theta(x)]\theta(x^2)+\theta(x^2)[T(x),\theta(x)]\}\theta(y)\{[T(x),\theta(x)]\theta(x^2)+\\ \theta(x^2)[T(x),\theta(x)]\}=0,\qquad \forall x,y\in R.\end{gathered}$$

It now follows that

$$[T(x),\theta(x)]\theta(x^2)+\theta(x^2)[T(x),\theta(x)]=0,\qquad \forall x\in R. \tag{24}$$

Replacing y by yx in (17) gives

$$\theta(x^3yx)[T(x),\theta(x)]-\theta(xyx^2)[T(x),\theta(x)]\theta(x)=0,\qquad \forall x,y\in R. \tag{25}$$

Replacing $\theta(y)$ by $[T(x),\theta(x)]\theta(y)$ in the above relation gives

$$\begin{aligned}&\theta(x^3)[T(x),\theta(x)]\theta(yx)[T(x),\theta(x)]\\ -\ &\theta(x)[T(x),\theta(x)]\theta(yx^2)[T(x),\theta(x)]\theta(x)=0,\qquad \forall x,y\in R.\end{aligned} \tag{26}$$

In the above relation let

$$\begin{gathered}a=\theta(x^3)[T(x),\theta(x)],b=\theta(x)[T(x),\theta(x)],c=-\theta(x^2)[T(x),\theta(x)]\theta(x)\text{ and}\\ z=\theta(y)\end{gathered}$$

From the above substitutions we have

$$azb+bzc=0.$$

We apply Lemma (2.2) to the above relation to obtain

$$\begin{aligned}\{\theta(x^3)[T(x),\theta(x)] \ &-\ \theta(x^2)[T(x),\theta(x)]\theta(x)\}\theta(yx)[T(x),\theta(x)]=0,\\ &\forall x,y\in R.\end{aligned} \tag{27}$$

Replacing y by yx^2 in the above relation gives

$$\begin{aligned}\{\theta(x^3)[T(x),\theta(x)] \ &-\ \theta(x^2)[T(x),\theta(x)]\theta(x)\}\theta(yx^3)[T(x),\theta(x)]=0,\\ &\forall x,y\in R.\end{aligned} \tag{28}$$

On the other hand replacing y by yx in relation (27) and right multiplying of this relation by $\theta(x)$ gives

$$\{\theta(x^3)[T(x),\theta(x)] - \theta(x^2)[T(x),\theta(x)]\theta(x)\}\theta(yx^2)[T(x),\theta(x)]\theta(x)=0, \quad \forall x,y\in R. \tag{29}$$

Subtracting (29) from (28) gives

$$\{\theta(x^3)[T(x),\theta(x)] - \theta(x^2)[T(x),\theta(x)]\theta(x)\}\theta(y)\{\theta(x^3)[T(x),\theta(x)] - \theta(x^2)[T(x),\theta(x)]\theta(x)\}=0, \qquad \forall x,y\in R.$$

It then follows that

$$\theta(x^3)[T(x),\theta(x)] - \theta(x^2)[T(x),\theta(x)]\theta(x)=0, \qquad \forall x\in R \tag{30}$$

Right multiplication of (24) by $\theta(x)$ gives

$$[T(x),\theta(x)]\theta(x^3)+\theta(x^2)[T(x),\theta(x)]\theta(x)=0, \qquad \forall x\in R. \tag{31}$$

According to (30) and (31) we have

$$[T(x),\theta(x)]\theta(x^3)+\theta(x^3)[T(x),\theta(x)]=0, \qquad \forall x\in R. \tag{32}$$

Left multiplication of (25) by $[T(x),\theta(x)]$ gives

$$[T(x),\theta(x)]\theta(x^3yx)[T(x),\theta(x)] - [T(x),\theta(x)]\theta(xyx^2)[T(x),\theta(x)]\theta(x)=0, \quad \forall x,y\in R. \tag{33}$$

Adding relations (26) and (33) and using (32) we obtain

$$\{[T(x),\theta(x)]\theta(x) + \theta(x)[T(x),\theta(x)]\}\theta(yx^2)[T(x),\theta(x)]\theta(x)=0, \quad \forall x,y\in R. \tag{34}$$

Using (30) we obtain from the above relation

$$\{[T(x),\theta(x)]\theta(x) + \theta(x)[T(x),\theta(x)]\}\theta(yx^3)[T(x),\theta(x)]=0, \quad \forall x,y\in R. \tag{35}$$

Left multiplication of (35) by $\theta(x^2)$ gives

$$\{\theta(x^2)[T(x),\theta(x)]\theta(x)+\theta(x^3)[T(x),\theta(x)]\}\theta(yx^3)[T(x),\theta(x)]=0, \qquad \forall x,y\in R.$$

According to (30) one can replace $\theta(x^2)[T(x),\theta(x)]\theta(x)$ by $\theta(x^3)[T(x),\theta(x)]$ in the above relation. Thus we have

$$\theta(x^3)[T(x),\theta(x)]\theta(y)\theta(x^3)[T(x),\theta(x)]=0, \qquad \forall x,y\in R.$$

It now follows that

$$\theta(x^3)[T(x),\theta(x)]=0, \qquad \forall x\in R. \tag{36}$$

Because of (32) we have

$$[T(x),\theta(x)]\theta(x^3)=0, \qquad \forall x\in R. \tag{37}$$

Replacing $\theta(y)$ by $[T(x),\theta(x)]\theta(y)$ in (17) gives

$$\begin{aligned}\theta(x^3)[T(x),\theta(x)]\theta(y)[T(x),\theta(x)] \quad - \quad & \theta(x)[T(x),\theta(x)]\theta(yx)[T(x),\theta(x)]\theta(x)\\ & = 0, \forall\ x,y\in R.\end{aligned} \tag{38}$$

Using (36) the above relation reduces to

$$\theta(x)[T(x),\theta(x)]\theta(yx)[T(x),\theta(x)]\theta(x)=0, \qquad \forall x,y\in R. \tag{39}$$

Replacing y by xy in (39) gives

$$\theta(x)[T(x),\theta(x)]\theta(x)\theta(y)\theta(x)[T(x),\theta(x)]\theta(x)=0, \qquad \forall x,y\in R,$$

It follows that

$$\theta(x)[T(x),\theta(x)]\theta(x)=0, \qquad \forall x\in R. \tag{40}$$

Putting $x+y$ for x in (40) we obtain

$$\begin{aligned} & \theta(x)[T(x),\theta(x)]\theta(y)+\theta(x)[T(x),\theta(y)]\theta(x)+\theta(x)[T(y),\theta(x)]\theta(x)\\ + \ & \theta(y)[T(x),\theta(x)]\theta(x)+\theta(x)[T(x),\theta(y)]\theta(y)+\theta(x)[T(y),\theta(x)]\theta(y)\\ + \ & \theta(y)[T(x),\theta(x)]\theta(y)+\theta(x)[T(y),\theta(y)]\theta(x)+\theta(y)[T(x),\theta(y)]\theta(x)\\ + \ & \theta(y)[T(y),\theta(x)]\theta(x)+\theta(x)[T(y),\theta(y)]\theta(y)+\theta(y)[T(x),\theta(y)]\theta(y)\\ + \ & \theta(y)[T(y),\theta(x)]\theta(y)+\theta(y)[T(y),\theta(y)]\theta(x)=0, \qquad \forall x,y\in R.\end{aligned} \tag{41}$$

Putting $-x$ for x in the above relation and combining the relation so obtained with (41) we obtain

$$\begin{aligned} & \theta(x)[T(x),\theta(y)]\theta(y)+\theta(x)[T(y),\theta(x)]\theta(y)\\ + \ & \theta(y)[T(x),\theta(x)]\theta(y)+\theta(x)[T(y),\theta(y)]\theta(x)\\ + \ & \theta(y)[T(x),\theta(y)]\theta(x)+\theta(y)[T(y),\theta(x)]\theta(x)=0, \forall x,y\in R.\end{aligned} \tag{42}$$

After comparing (41) and (42)we have

$$\begin{aligned} & \theta(x)[T(x),\theta(x)]\theta(y)+\theta(x)[T(x),\theta(y)]\theta(x)+\theta(x)[T(y),\theta(x)]\theta(x)\\ + \ & \theta(y)[T(x),\theta(x)]\theta(x)+\theta(x)[T(y),\theta(y)]\theta(y)+\theta(y)[T(x),\theta(y)]\theta(y)\\ + \ & \theta(y)[T(y),\theta(x)]\theta(y)+\theta(y)[T(y),\theta(y)]\theta(x)=0, \qquad \forall x,y\in R.\end{aligned} \tag{43}$$

Replacing x by $2x$ in the above relation and subtracting the relation so obtained from the above relation multiplied by 8, we obtain

$$\begin{aligned} \theta(x)[T(y),\theta(y)]\theta(y) &+ \theta(y)[T(x),\theta(y)]\theta(y)+\theta(y)[T(y),\theta(x)]\theta(y) \\ &+ \theta(y)[T(y),\theta(y)]\theta(x)=0, \qquad \forall x,y\in R. \end{aligned} \tag{44}$$

Comparing (43) and (44) we obtain

$$\begin{aligned} \theta(x)[T(x),\theta(x)]\theta(y) &+ \theta(x)[T(x),\theta(y)]\theta(x)+\theta(x)[T(y),\theta(x)]\theta(x) \\ &+ \theta(y)[T(x),\theta(x)]\theta(x)=0, \qquad \forall x,y\in R. \end{aligned} \tag{45}$$

Right multiplication of (45) by $\theta(x^2)[T(x),\theta(x)]$ and using (36) gives

$$\theta(x)[T(x),\theta(x)]\theta(y)\theta(x^2)[T(x),\theta(x)]=0, \qquad \forall x,y\in R. \tag{46}$$

Left multiplication of (46) by $\theta(x)$ gives

$$\theta(x^2)[T(x),\theta(x)]\theta(y)\theta(x^2)[T(x),\theta(x)]=0, \qquad \forall x,y\in R.$$

It follows, by the surjectivity of θ and the semiprimeness of R, that

$$\theta(x^2)[T(x),\theta(x)]=0, \qquad \forall x\in R. \tag{47}$$

Because of (24) we also have

$$[T(x),\theta(x)]\theta(x^2)=0, \qquad \forall x\in R. \tag{48}$$

Right multiplication of (45) by $\theta(x)[T(x),\theta(x)]$ gives because of (47)

$$\theta(x)[T(x),\theta(x)]\theta(y)\theta(x)[T(x),\theta(x)]=0, \qquad \forall x,y\in R.$$

By the surjectivity of θ and the semiprimeness of R it follows that

$$\theta(x)[T(x),\theta(x)]=0, \qquad \forall x\in R. \tag{49}$$

Left multiplication of (45) by $[T(x),\theta(x)]\theta(x)$ and use of (48) gives

$$[T(x),\theta(x)]\theta(x)\theta(y)[T(x),\theta(x)]\theta(x)=0, \qquad \forall x,y\in R.$$

By the surjectivity of θ and the semiprimeness of R we obtain

$$[T(x),\theta(x)]\theta(x)=0, \qquad \forall x\in R. \tag{50}$$

Replacing x by $x+y$ in (50) and then using (50) gives

$$[T(x),\theta(x)]\theta(y)+[T(x),\theta(y)]\theta(x)+[T(x),\theta(y)]\theta(y)+[T(y),\theta(x)]\theta(x)+ \\ [T(y),\theta(x)]\theta(y)+[T(y),\theta(y)]\theta(x)=0, \quad \forall\, x,y\in R.$$

Putting $-x$ for x in the above relation and comparing the relation so obtained with the above relation gives:

$$[T(x),\theta(y)]\theta(x) + [T(y),\theta(x)]\theta(x) + [T(x),\theta(x)]\theta(y) = 0, \quad \forall\, x,y \in R. \tag{51}$$

Right multiplication of the above relation by $[T(x),\theta(x)]$ use of (49) gives

$$[T(x),\theta(x)]\theta(y)[T(x),\theta(x)] = 0, \quad \forall\, x,y \in R.$$

By the surjectivity of θ and the semiprimeness of R we get,

$$[T(x),\theta(x)] = 0, \quad \forall\, x \in R. \tag{52}$$

Now we will prove that

$$T(xy+yx) = T(y)\theta(x) + \theta(x)T(y), \quad \forall\, x,y \in R. \tag{53}$$

In order to prove the above relation we need to prove the following relation

$$[A(x,y),\theta(x)] = 0, \quad \forall\, x,y \in R, \tag{54}$$

where $A(x,y)$ stands for $T(xy+yx) - T(y)\theta(x) - \theta(x)T(y)$. With respect to this notation equation (2) can be rewritten as,

$$\theta(x)A(x,y)\theta(x) = 0, \quad \forall\, x,y \in R. \tag{55}$$

Replacing x by $x+y$ in relation (50) gives

$$[T(x),\theta(y)] + [T(y),\theta(x)] = 0, \quad \forall\, x,y \in R. \tag{56}$$

After replacing y by $xy+yx$ in (56) and using (50) we obtain

$$\theta(x)[T(x),\theta(y)] + [T(x),\theta(y)]\theta(x) + [T(xy+yx),\theta(x)] = 0, \quad \forall\, x,y \in R.$$

According to (56) we can replace in the above relation $[T(x),\theta(y)]$ by $-[T(y),\theta(x)]$. We then have

$$[T(xy+yx),\theta(x)] - \theta(x)[T(y),\theta(x)] - [T(y),\theta(x)]\theta(x) = 0, \quad \forall\, x,y \in R.$$

This can be written in the form

$$[T(xy+yx) - T(y)\theta(x) - \theta(x)T(y),\theta(x)] = 0, \quad \forall\, x,y \in R.$$

The proof of relation (54) is therefore complete.
Replacing x by $x+z$ in (55) and using (55) gives

$$\theta(x)A(x,y)\theta(z) + \theta(x)A(z,y)\theta(x) + \theta(z)A(x,y)\theta(x) + \theta(z)A(z,y)\theta(x) + \theta(z)A(x,y)\theta(z) + \theta(x)A(z,y)\theta(z) = 0, \quad \forall\, x,y,z \in R.$$

After replacing x for $-x$ in the above relation and adding the relation so obtained to the above relation we arrive at:

$$\theta(x)A(x,y)\theta(z)+\theta(x)A(z,y)\theta(x)+\theta(z)A(x,y)\theta(x)=0, \quad \forall\, x,y,z\in R.$$

Right multiplication of the above relation by $A(x,y)\theta(x)$ and using (55) gives

$$\theta(x)A(x,y)\theta(z)A(x,y)\theta(x)=0, \quad \forall\, x,y,z\in R. \tag{57}$$

Using (54), the above relation can be written in the form

$$\theta(x)A(x,y)\theta(z)\theta(x)A(x,y)=0, \quad \forall\, x,y,z\in R. \tag{58}$$

By the surjectivity of θ and the semiprimeness of R it follows that

$$\theta(x)A(x,y)=0, \quad \forall\, x,y\in R. \tag{59}$$

From (54) and (59) we also get

$$A(x,y)\theta(x)=0, \quad \forall\, x,y\in R. \tag{60}$$

Replacing x by $x+z$ in (60) gives

$$A(x,y)\theta(z)+A(z,y)\theta(x)=0, \quad \forall\, x,y,z\in R.$$

Right multiplication of the above relation by $A(x,y)$ and using (59) gives

$$A(x,y)\theta(z)A(x,y)=0, \quad \forall\, x,y,z\in R,$$

By the surjectivity of θ and the semiprimeness of R it follows that

$$A(x,y)=0, \quad \forall\, x,y\in R.$$

The proof of (53) is therefore complete. In particular when $y=x$ (53) reduces to

$$2T(x^2)=T(x)\theta(x)+\theta(x)T(x), \quad \forall\, x\in R.$$

Using (52), the above relation yields

$$T(x^2)=T(x)\theta(x), \quad \forall\, x\in R,$$

and

$$T(x^2)=\theta(x)T(x), \quad \forall\, x\in R,$$

This means that T is a left and also a right Jordan $\theta-$centralizer. By Theorem (2) in [1] it follows that T is a left and also right $\theta-$centralizer, which completes the proof. □

As a direct consequence, when θ is the identity map then Theorem 2.3.2 in [2] gives as the following result

Corollary 2.3. *Let R be a 2-torsion free semiprime ring and let $T : R \longrightarrow R$ be an additive mapping. suppose that $T(xyx) = T(x)yx + xT(y)x + xyT(x)$ and $xT(xy + yx)x = xT(y)x^2 + x^2T(y)x$ holds for all pairs $x, y \in R$. Then T is a left and also a right centralizer of R. Moreover, there exist an element $\lambda \in C$, where C is the extended centroid of R, such that $T(x) = \lambda x$ for all $x \in R$.*

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