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Note on a New Kind of

Hardy-Hilbert's Integral Inequality

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Abstract

Some new kind of inequalities similar to Hardy-Hilbert's inequality are given.

1. Introduction

If $f, g \geq 0$ such that

$$0 < \int_0^\infty f^2(x)\,dx < \infty, \quad 0 < \int_0^\infty g^2(x)\,dx < \infty\ ,$$

then the famous Hilbert's integral inequality is given by

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy < \pi\left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(x)\,dx\right)^{1/2}. \tag{1}$$

where the constant factor π is the best possible (see[2]) . Inequality (1) has been generalized by Hardy-Riesz [1] as

If p>1, $\frac{1}{p}+\frac{1}{q}=1$,

$$0 < \int_0^\infty f^p(x)\,dx < \infty\ , \quad 0 < \int_0^\infty g^q(x)\,dx < \infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty g^q(x)\,dx\right)^{1/q}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. We call (2) Hardy-Hilbert's integral inequality, which is important in analysis and its application (see[3]).
Yang [4,5] has extended inequality (2) by proving the following

If $f, g \geq 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \lambda > 2 - \min\{p, q\},$ such that

$$0 < \int_0^\infty x^{1-\lambda} f^p(x)\, dx < \infty, \quad 0 < \int_0^\infty x^{1-\lambda} g^q(x)\, dx < \infty\ ,$$

then the extended Hardy-Hilbert's inequality is given by

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{(x+y)^\lambda} dx\, dy < k_\lambda(p) \left(\int_0^\infty t^{1-\lambda} f^p(t)\, dt \right)^{1/p} \left(\int_0^\infty t^{1-\lambda} g^q(t)\, dt \right)^{1/q}, \tag{3}$$

where the constant $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right),$ is the best possible, B is the beta function.

The aim of this paper is to give new kinds of Hardy-Hilbert's integral inequality

2. New Inequalities

In what follows we may write f, F, g ,G instead of f(x), F(x), g(y), G(y) .
The following lemma is needed for our aim
Lemma [2, Th.342 Remark] .

$$\int_0^\infty \frac{\ln u}{u-1} u^{-\frac{1}{r}}\, du = \left(\frac{\pi}{\sin(\pi / r)} \right)^2, \quad r > 1.$$

We state and prove the following

Theorem 1. *Let* $f, g, F, G \geq 0,\ F', G' > 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \lambda > 0 . r > 1,$ $F(0) = G(0) = 0,\ \ F(\infty), G(\infty)\ \leq \infty.$ *Then, we have*

$$\int_0^\infty \int_0^\infty \frac{\ln\left(\frac{F}{G} \right) f\ g}{F^\lambda - G^\lambda} dx\, dy \leq \left(\frac{\pi}{\lambda \sin(\pi / r)} \right)^2 \left(\int_0^\infty \frac{F^{p\left(\frac{\lambda}{r} - \lambda + 1 \right) - 1} f^p}{(F')^{p-1}} dx \right)^{1/p} \times \left(\int_0^\infty \frac{G^{q\left(\frac{\lambda}{r} - \lambda + 1 \right) - 1} g^q}{(G')^{q-1}} dy \right)^{1/q}, \tag{4}$$

$$\int_0^\infty G^{(1-p)\left[q\left(\frac{\lambda}{r} \lambda - +1 \right) - 1 \right]} G' \left(\int_0^\infty \frac{\ln\left(\frac{F}{G} \right) f}{F^\lambda - G^\lambda} dx \right)^p dy \leq \left(\frac{\pi}{\lambda \sin(\pi / r)} \right)^{2p} \int_0^\infty \frac{F^{p\left(\frac{\lambda}{r} - \lambda + 1 \right) - 1} f^p}{(F')^{p-1}} dx . \tag{5}$$

The inequalities (4) and (5) are equivalent.

Proof. We have

$$\int_0^\infty\int_0^\infty \frac{\ln\left(\frac{F}{G}\right) f\, g}{F^\lambda - G^\lambda}\, dx\, dy$$

$$= \int_0^\infty\int_0^\infty \frac{\ln^{1/p}\left(\frac{F}{G}\right) f\, G^{\frac{1}{p}\left(\lambda-1-\frac{\lambda}{r}\right)} (G')^{1/p}}{(F')^{1/q} F^{\frac{1}{q}\left(\lambda-1-\frac{\lambda}{r}\right)} \left(F^\lambda - G^\lambda\right)^{1/p}} \times \frac{\ln^{1/q}\left(\frac{F}{G}\right) g\, F^{\frac{1}{q}\left(\lambda-1-\frac{\lambda}{r}\right)} (F')^{1/q}}{(G')^{1/p} G^{\frac{1}{p}\left(\lambda-1-\frac{\lambda}{r}\right)} \left(F^\lambda - G^\lambda\right)^{1/q}}\, dx\, dy$$

$$\le \left(\int_0^\infty\int_0^\infty \frac{\ln\left(\frac{F}{G}\right) f^p\, G^{\lambda-1-\lambda/r}\, G'}{(F')^{p/q} F^{\frac{p}{q}(\lambda-1-\lambda/r)} \left(F^\lambda - G^\lambda\right)}\, dx\right)^{1/p} \left(\int_0^\infty\int_0^\infty \frac{\ln\left(\frac{F}{G}\right) g^q\, F^{\lambda-1-\lambda/r} F'}{(G')^{q/p} G^{\frac{q}{p}(\lambda-1-\lambda/r)} \left(F^\lambda - G^\lambda\right)}\, dy\right)^{1/q}$$

$$= M^{1/p} N^{1/q}.$$

$$M = \frac{1}{\lambda^2} \int_0^\infty \frac{F^{\left(1+\frac{\lambda}{r}-\lambda\right)-\frac{1}{r}} f^p}{(F')^{p/q}}\, dx \int_0^\infty \frac{\ln\left(\frac{F}{G}\right)^\lambda \lambda \left(\frac{G}{F}\right)^{\lambda-1-\lambda/r} \frac{G'}{F}}{\left(\frac{G}{F}\right)^\lambda - 1}\, dy$$

$$\le \frac{1}{\lambda^2} \int_0^\infty \frac{F^{p\left(\frac{\lambda}{r}-\lambda+1\right)-1} f^p}{(F')^{p-1}}\, dx \int_0^\infty \frac{\ln u\; u^{-\frac{1}{r}}}{u-1}\, du$$

$$= \left(\frac{\pi}{\lambda \sin(\pi/r)}\right)^2 \int_0^\infty \frac{F^{p\left(\frac{\lambda}{r}-\lambda+1\right)-1} f^p}{(F')^{p-1}}\, dx.$$

Similarly,

$$N \le \left(\frac{\pi}{\lambda \sin(\pi/r)}\right)^2 \int_0^\infty \frac{G^{q\left(\frac{\lambda}{r}-\lambda+1\right)-1} g^q}{(G')^{q-1}}\, dy.$$

In order to show that the inequalities (4) and (5) are equivalent, suppose that (4) holds, then, we have

$$\int_0^\infty G^{(1-p)\left[q\left(\frac{\lambda}{r}-\lambda+1\right)-1\right]} G' \left(\int_0^\infty \frac{\ln\left(\frac{F}{G}\right) f}{F^{\lambda} - G^{\lambda}}\,dx\right)^{p} dy$$

$$= \int_0^\infty\int_0^\infty \frac{\ln\left(\frac{F}{G}\right) f\, G^{(1-p)\left[q\left(\frac{\lambda}{r}-\lambda+1\right)-1\right]} G' \left(\int_0^\infty \frac{\ln\left(\frac{F}{G}\right) f}{F^{\lambda} - G^{\lambda}}\,dx\right)^{p-1}}{F^{\lambda} - G^{\lambda}}\,dx\,dy$$

$$\le \left(\frac{\pi}{\lambda \sin(\pi/r)}\right)^{2} \left(\int_0^\infty \frac{F^{p\left(\frac{\lambda}{r}-\lambda+1\right)-1} f^{p}}{(F')^{p-1}}\,dx\right)^{1/p} \times$$

$$\left(\int_0^\infty \frac{G^{q\left(\frac{\lambda}{r}-\lambda+1\right)-1} G^{q(1-p)\left[q\left(\frac{\lambda}{r}-\lambda+1\right)-1\right]} (G')^{q} \left(\int_0^\infty \frac{\ln\left(\frac{F}{G}\right) f}{F^{\lambda} - G^{\lambda}}\right)^{q(p-1)}}{(G')^{q-1}}\,dy\right)^{1/q}$$

$$= \left(\frac{\pi}{\lambda \sin(\pi/r)}\right)^{2} \left(\int_0^\infty \frac{F^{p\left(\frac{\lambda}{r}-\lambda+1\right)-1} f^{p}}{(F')^{p-1}}\,dx\right)^{1/p} \times$$

$$\left(\int_0^\infty G^{(1-p)\left[q\left(\frac{\lambda}{r}-\lambda+1\right)\right]} G' \left(\int_0^\infty \frac{\ln\left(\frac{F}{G}\right) f}{F^{\lambda} - G^{\lambda}}\,dx\right)^{p} dy\right)^{1/q}$$

which implies

$$\left(\int_0^\infty G^{(1-p)\left[q\left(\frac{\lambda}{r}-\lambda+1\right)-1\right]} G' \left(\int_0^\infty \frac{\ln\left(\frac{F}{G}\right) f}{F^{\lambda} - G^{\lambda}}\,dx\right)^{p} dy\right)^{1/p} \le \left(\frac{\pi}{\lambda \sin(\pi/r)}\right)^{2} \times$$

$$\left(\int_0^\infty \frac{F^{p\left(\frac{\lambda}{r}-\lambda+1\right)-1} f^p}{(F')^{p-1}} dx \right)^{1/p},$$

and hence (5) is obtained.
Now suppose (5) is satisfied, then

$$\int_0^\infty \int_0^\infty \frac{\ln\left(\frac{F}{G}\right) f\, g}{F^\lambda - G^\lambda} dx\, dy$$

$$= \int_0^\infty G^{-\frac{1}{p}(1-p)\left[q\left(\frac{\lambda}{r}-\lambda+1\right)-1\right]} (G')^{-\frac{1}{p}} g \left(G^{\frac{1}{p}(1-p)\left[q\left(\frac{\lambda}{r}-\lambda+1\right)-1\right]} (G')^{\frac{1}{p}} \left(\int_0^\infty \frac{\ln\left(\frac{F}{g}\right) f}{F^\lambda - G^\lambda} dx \right)\right) dy$$

$$\leq \left(\int_0^\infty G^{\frac{q}{p}(p-1)\left[q\left(\frac{\lambda}{r}-\lambda+1\right)\right]} (G')^{-\frac{q}{p}} g^q\, dy \right)^{1/q} \times$$

$$\left(\int_0^\infty G^{(1-p)\left[p\left(\frac{\lambda}{r}-\lambda+1\right)\right]} G' \left(\int_0^\infty \frac{\ln\left(\frac{F}{G}\right) f}{F^\lambda - G^\lambda} dx \right)^p dy \right)^{1/p}$$

$$\leq \left(\frac{\pi}{\lambda \sin(\pi / r)} \right)^2 \left(\int_0^\infty \frac{F^{p\left(\frac{\lambda}{r}-\lambda+1\right)-1} f^p}{(F')^{p-1}} dx \right)^{1/p} \left(\int_0^\infty \frac{G^q g^{q\left(\frac{\lambda}{r}-\lambda+1\right)-1}}{(G')^{q-1}} dy \right)^{1/q}.$$

Completes the proof.

Corollary 2. *Let* f, g, p, q, λ, r *be as defined in theorem*1. *Let* $F(\infty), G(\infty) \leq \infty$, *where* F *and* G *are defined by*

$$F(t) = \int_0^t f(u)\, du, \qquad G(t) = \int_0^t g(u)\, du.$$

Then

$$\text{(6)} \qquad \int_0^\infty \int_0^\infty \frac{\ln\left(\frac{F}{G}\right) f\, g}{F^\lambda - G^\lambda} dxdy \leq \left(\frac{\pi}{\lambda \sin(\pi / r)} \right)^2 \left(\int_0^\infty f\, F^{p\left(\frac{\lambda}{r}-\lambda+1\right)-1} dx \right)^{1/p} \times$$

$$\left(\int_0^\infty g \, G^{q\left(\frac{\lambda}{r} - \lambda + 1\right) - 1} dy \right)^{1/q} .$$

Proof. Follows from theorem 1, noticing that $F' = f$, $G' = g$.

Corollary 3. *Let the assumption of theorem* 1 *is satisfied. Then*

$$(7) \qquad \int_0^\infty \int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f \, g}{x^\lambda - y^\lambda} dx \, dy \le \left(\frac{\pi}{\lambda \sin(\pi / r)} \right)^2 \left(\int_0^\infty x^{p\left(\frac{\lambda}{r} - \lambda + 1\right) - 1} f^p \, dx \right)^{1/p} \times \left(\int_0^\infty y^{q\left(\frac{\lambda}{r} - \lambda + 1\right) - 1} g^q \, dy \right)^{1/q} .$$

Proof. Follows from theorem 1, by putting F(t) = G(t) = t .

References

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