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Derivation into Duals of Ideals of C^*-Algebras

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Abstract

Let A be a C^*-algebra and I be a closed ideal of A. we show that A is n-I-weakly amenable for each $n \in \mathbb{N}$ i.e. $H^1(A, I^{(n)}) = 0$. This result generalize the fact that asserts all C^*-algebras are ideally amenable.

Keywords: Amenability, C^*-algebra, derivation, ideally amenable

1 Introduction

Let A be a Banach algebra and let X be a Banach A-bimodule. Then X' is a also a Banach A-bimodule. Indeed the module action is a defined by

$$\langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle \qquad (a \in A, \lambda \in X', x \in X)$$

$$\langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \qquad (a \in A, \lambda \in X', x \in X)$$

Similarly, the higher duals $X^{(n)}$ are Banach A-bimodules. If the canonical embedding of X in X'' is denoted by $\hat{}$ (or τ) then $a \cdot \hat{x} = \widehat{a \cdot x}$, so that $X^{(n)}$ is a submodule of $X^{(n+2)}$ for each $n \in \mathbb{Z}$. (when no ambiguity seems possible, we shall omit $\hat{}$).

For any $n \in \mathbb{N}$, the adjoint of the injective map $\tau : X^{(n-1)} \to X^{(n+1)}$ is the projection $P : X^{(n+2)} \to X^{(n)}$, defined by $P(\Lambda) = \Lambda|_{\tau(X^{(n-1)})}$. Then P is a A-bimodule homomorphism and we may write

$$X^{(n+2)} = X^{(n)} \oplus \ker P = X^{(n)} \oplus \tau(X^{(n-1)})^{\perp}$$

A derivation from A into X is a linear map D such that $D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A)$.

For example, let $x \in X$ then $\delta_x : A \to X$, $\delta_x(a) = a \cdot x - x \cdot a \quad (a \in A)$ is a derivation that is called inner derivation. The space of continuous derivations from A into X is denoted by $Z^1(A, X)$ and the space of inner derivations from A into X is denoted by $N^1(A, X)$. The first cohomology group of A with coefficients in X is defined by

$$H^1(A, X) = \frac{Z^1(A, X)}{N^1(A, X)}$$

The Banach algebra A is amenable if $H^1(A, X') = 0$ for each Banach A-bimodule X; and it is weakly amenable if $H^1(A, A') = 0$. Let A be a Banach algebra and let I be a closed ideal of A, A is I-weakly amenable if $H^1(A, I') = 0$. A is n-I-weakly amenable if $H^1(A, I^{(n)}) = 0$. A is ideally amenable if A is I-weakly amenable for every closed ideal I in A.

For example, it was shown in [4] that group algebra $L^1(G)$ is amenable if and only if G is an amenable group, and in [5] that $L^1(G)$ is weakly amenable for each locally compact group G. We know that every C^*-algebra is ideally amenable [3], and also is permantly weakly amenable that is $H^1(A, A^{(n)}) = 0$ for each $n \in \mathbb{N}$. Our purpose in this paper is to show that $H^1(A, I^{(n)}) = 0$ for each $n \in \mathbb{N}$, where A is a C^*-algebra, and I is a closed ideal of A.

2 Preliminaries

Let A be a Banach algebra. Then A'' is a Banach algebra; indeed, two products are defined on A'' as follows.

$$\langle a, \Phi \cdot \lambda \rangle = \langle \lambda \cdot a, \Phi \rangle \quad , \quad \langle a, \lambda \cdot \Phi \rangle = \langle a \cdot \lambda, \Phi \rangle$$

$$\langle \lambda, \Phi \Box \Psi \rangle = \langle \Psi \cdot \lambda, \Phi \rangle \quad , \quad \langle \lambda, \Phi \Diamond \Psi \rangle = \langle \lambda \cdot \Phi, \Psi \rangle$$

where $a \in A$, $\lambda \in A'$ and $\Phi, \Psi \in A''$.

Each of the products $\Box$ and $\Diamond$ make A'' Banach algebra, these products are called the first and second *Arens products* on A'', respectively. A is *Arens regular* if $\Box$ and $\Diamond$ coincide on A''.

The products $\Box$ and $\Diamond$ both extend the module operation A'' i.e.

$$a \cdot \Phi = \hat{a} \Box \Phi = \hat{a} \Diamond \Phi \quad , \quad \Phi \cdot a = \Phi \Box \hat{a} = \Phi \Diamond \hat{a} \qquad (a \in A, \Phi \in A'')$$

Let A be a Banach algebra and let I be a closed ideal of A then I'' is a closed ideal of $(A'', \Box)$. Since $i : I'' \to A''$ is continuous with closed rang in A'', the topologies $\sigma(A'', A)$ and $\sigma(I'', I)$ on I'' are equivalent. Now let A be a Banach algebra, and let X be a Banach A-bimodule. Then X'' is a Banach A''-bimodule. Indeed

$$\Lambda \cdot \Phi = \lim_{\delta} \lim_{\gamma} x_\delta \cdot a_\gamma \qquad \text{in } (X'', \sigma(X'', X'))$$

$$\Phi \cdot \Lambda = \lim_{\gamma} \lim_{\delta} a_\gamma \cdot x_\delta \qquad \text{in } (X'', \sigma(X'', X'))$$

where $\Lambda \in X''$, $\Phi \in A''$, $a_\gamma \to \Phi$, and $x_\delta \to \Lambda$ in $(A'', \sigma(A'', A'))$ and $(X'', \sigma(X'', X'))$, respectively.

As in [2] we have the following theorems.

Theorem 2.1 *Let A be a Banach algebra and let X be a Banach A-bimodule. Suppose that $D : A \to X$ is a continuous derivation. Then $D'' : (A'', \Box) \to X''$ is a continuous derivation.*

Theorem 2.2 *Let A be a Banach algebra and let X be a Banach A-bimodule.*

(i) The map $P : X'''' \to X''$ is a left $(A'', \Box)$-module homomorphism.

(ii) Suppose that the map $\Phi \to \lambda \cdot \Phi$, $(A'', \sigma(A'', A')) \to (X'', \sigma(X'', X'))$

is continuous for each fixed $\lambda \in X''$. Then P is a also a right $(A'', \Box)$-module homomorphism.

Note that, in the case where $X = I$ in the above result the extra hypothesis in (ii) is obtained by Arens regularity of A. For it, the map $\Phi \to \lambda \cdot \Phi (= \lambda \Box \Phi)$ is $\sigma(A'', A') \to \sigma(A'', A')$ continuous, and also on I'' the topologies $\sigma(A'', A')$ and $\sigma(I'', I')$ are coincide.

Theorem 2.3 *Let A be a Banach algebra, let $n \in \mathbb{N}$, and let $D : A \to I^{(n)}$ be a continuous derivation, where I is a closed ideal of A. Suppose $A^{(2n-2)}$ is Arens regular. Then there is a continuous derivation $\widetilde{D} : (A^{(2n)}, \Box) \to (I^{(2n)}, \Box)$ such that $\widetilde{D}(\tilde{a}) = D(a) \quad (a \in A)$. Where $\tilde{a}$ is the canonical image in $A^{(2n)}$ of $a \in A$.*

Proof. By Theorem 2.1. $D'' : (A'', \Box) \to I^{(2n+2)}$ is a continuous derivation. Set $B = I^{(2n-2)}$, since $A^{(2n-2)}$ is Arens regular, by Theorem 2.2. the canonical projection $P : B'''' \to B''$ (Indeed $P : I^{(2n+2)} \to I^{(2n)}$) is a $A^{(2n)}$-bimodule homomorphism.

$$D'' : A'' \to I^{(2n+2)} \qquad , \qquad P : I^{(2n+2)} \to I^{(2n)}$$

Define $\widetilde{D} = PD''$. Since D'' is a derivation and P is A''-bimodule homomorphism, $\widetilde{D} : A'' \to I^{(2n)}$ is a derivation. When $n = 1$ the above argument is sufficient to prove theorem. In general we use induction. Suppose conclusion is satisfied for $n-1$. We prove for n. As preceding argument there is a derivation $\widetilde{D}_1 : A'' \to (I'')^{(2n)}$ that extend D.
Now $\widetilde{D}_1 : A'' \to (I'')^{2(n-1)}$ is a continuous derivation, I'' is a closed ideal of A'' and $I^{(2n)}$ is $2(n-1)$-dual of I''. Hence by induction hypothesis there is a continuous derivation $\widetilde{D} : (A'')^{(2n-1)} \to (I'')^{2(n-1)}$ (Indeed $\widetilde{D} : A^{(2n)} \to I^{(2n)}$). Thus induction argument is completed.■

We need the following facts about weakly compact maps.

Theorem 2.4 *The following are equivalent for a continuous linear map $T : E \to F$.*

(i) T is weakly compact.

(ii) $T''(E'') \subset \tau(F)$ (where $\tau : F \to F''$ is canonical embedding). [1;A.3.56]

Theorem 2.5 (Akeman.) *Let A and B be C^*-algebras. Then each continuous linear operator from A into B' is weakly compact. [1;3.2.48]*

In the end of this section, we recall the following result of Sakai [6;4.1.8].

Theorem 2.6 *Let A be a Von Neumann algebra. Then $H^1(A, A) = 0$.*

3 The main Theorem

By the preceding result of Sakai we have the following Lemma.

Lemma 3.1 *Let A be a Von Neumann algebra and let I be a closed ideal of A that is also a Von Neumann algebra. Then $H^1(A, I) = 0$.*

Proof. Let $D : A \to I$ be a continuous derivation. Then $D|_I : I \to I$ is a continuous derivation. Since I is a von Neumann algebra, by Theorem 2.5. we have $D|_I \in N(I, I)$ i.e. there is $x \in I$ such $D|_I = \delta_x$. Thus $D(a) = ax - xa$

for each $a \in I$. Since I is a Von Neumann algebra, it has a unit, say e.
Now for each $b \in A$ we have:

$$D(be) = D|_I(be) = bex - xbe = bx - xb$$

On the other hand $D(be) = D(b)e + bD(e) = D(b)$. Thus $D(b) = bx - xb$ and D is inner.■

Theorem 3.1 *let I be a closed ideal of C^*-algebra A. Then A is n-I-weakly amenable for each $n \in \mathbb{N}$.*

Proof. First we show that $H^1(A, I^{(2n)}) = 0$. Let $D : A \to I^{(2n)}$ be a continuous derivation. By Theorem 2.3 there is a continuous derivation $\widetilde{D} : A^{(2n)} \to I^{(2n)}$ that extends D. Since $A^{(2n)}$ and $I^{(2n)}$ are von Neumann algebra, and $I^{(2n)}$ is a closed ideal of $A^{(2n)}$, by Lemma 3.1. $H^1(A^{(2n)}, I^{(2n)}) = 0$. Thus $\widetilde{D}$ is inner and so D.
We now show that A is $(2n+1)$-I-weakly amenable, under the assumption that B is $(2n-1)$-J-weakly amenable for every C^*-algebra B and closed ideal J of B. (it is sufficient by induction).
Let $D \in Z^1(A, I^{(2n+1)})$. Since $I^{(2n)}$ is a C^*-algebra, by Theorem 2.5. D is weakly compact and so by Theorem 2.4. $D''(A'') \subseteq I^{(2n+1)}$. Hence $D'' : A'' \to (I'')^{(2n-1)}$ is continuous derivation and by the assumption, D'' is inner i.e. there is $\Lambda \in I^{(2n+1)}$ such that

$$D''(\Phi) = \Phi \Box \Lambda - \Lambda \Box \Phi \qquad (\Phi \in A'')$$

In particular $D(a) = a \cdot \Lambda - \Lambda \cdot a \quad (a \in A)$.
Not that the case where $n = 1$ is proved in [3].■

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