The $q$-Numerical Range of a Certain $3 \times 3$ Matrix

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Abstract
The paper provides the equation of the boundary of the $q$-numerical range of a certain $3 \times 3$ unitarily irreducible matrix.

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1. Introduction and the main result

In this decade, some authors have dealt with the $q$-numerical range of a matrix and have obtained interesting results (cf.[8],[9],[10],[4],[5]). Let $A$ be an $n \times n$ complex matrix and $0 \leq q \leq 1$. The $q$-numerical range of $A$ is defined and denoted by

$$W_q(A) = \{\zeta^* A \xi : \xi, \zeta \in \mathbb{C}^n, \xi^* \xi = 1, \zeta^* \xi = q\}.$$ (1.1)

In [12] Tsing shows that the range $W_q(A)$ satisfies the formula

$$W_q(A) = \{q \xi^* A \xi + \sqrt{1 - q^2 w} \sqrt{\xi^* A^* A \xi - \xi^* A \xi} : w \in \mathbb{C}, |w| \leq 1\}$$
\[ \xi \in \mathbb{C}^n, \xi^*\xi = 1, \]

and using this formula he shows the range \( W_q(A) \) is convex. The \( q \)-numerical range is unitary similarity invariant, \( W_q(A) = W_q(UAU^*) \), for any unitary matrix \( U \). It is also transpose invariant, \( W_q(A) = W_q(A^T) \). From these viewpoints, the set \( W_q(A) \) is a natural generalization of the classical numerical range \( W(A) = W_1(A) \) which was introduced by Toeplitz [12]. As a consequence of Tarski-Seidenberg theorem (cf.[2]), the boundary of the set \( W_q(A) \) lies on an algebraic curve. The range \( W_q(A) \) is closely related with the following homogeneous polynomial:

\[
F_A(X,Y,Z,W) = \det(WI_n + X/2(A + A^*) - iY/2(A - A^*) + ZA^*A).
\]

If the same size square matrices \( A, B \) satisfies \( F_A = F_B \), then the equation \( W_q(A) = W_q(B) \) holds for \( 0 \leq q \leq 1 \). In [10], [4], it is shown that the range \( W_q(A) \) is determined by the set

\[
W(A, A^*A) = \{(\xi^*A\xi, \xi^*A^*A\xi) \in \mathbb{C} \times \mathbb{R} : \xi \in \mathbb{C}^n, \xi^*\xi = 1\}.
\]

Especially, if \( A, B \) are respective \( n \times n \), \( m \times m \) matrices satisfying \( W(A, A^*A) = W(B, B^*B) \), then the equation \( W_q(A) = W_q(B) \) holds for \( 0 \leq q \leq 1 \). The set \( W(A, A^*A) \) is called the Davis-Wielandt shell for \( A \) (cf. [4]). By results of [1] or [3], the Davis-Wielandt shell \( W(A, A^*A) \) is convex for \( n \geq 3 \). If \( n = 2 \), its boundary surrounds a convex region. If same size square matrices \( A, B \) satisfies the equation

\[
F_A(X,Y,Z,1) = F_B(X,Y,Z,1)
\]

for every \((X,Y,Z) \in \mathbb{R}^3\), then \( W(A, A^*A) = W(B, B^*B) \). If \( A \) is an \( n \times n \) matrix, then the boundary of \( W(A, A^*A) \) lies on an algebraic surface of order less than or equal to \( n(n-1)^2 \).

By using Tsing’s circular union formula (1.2), we can perform a numerical approximation for a rather wide class of matrices \( A \). The equation of the boundary of \( W_q(A) \) for a \( 3 \times 3 \) unitarily reducible matrix \( A \) is known (cf. [7],[5],[6]). However the equation of the boundary of \( W_q(A) \) for a unitarily irreducible \( 3 \times 3 \) matrix \( A \) is not known yet. We will provide a typical example of such an object. Let

\[
N = \begin{pmatrix}
0 & 1 & 1/2 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\]
We obtain the exact equation of $\partial W_q(N)$. This example provides a starting point to the general study of the $q$-numerical of unitarily irreducible $3 \times 3$ matrices.

**Theorem 1.1** Suppose that $N$ is a $3 \times 3$ nilpotent matrix given by (1.5) and $q$ is a a real number $0 \leq q \leq 1$. (1) If $q = 1$, then the boundary of $W_1(N)$ lies on the sextic algebraic curve

$$\{x + iy : (x, y) \in \mathbb{R}^2, 11664x^6 + 28080x^4y^2 + 21168x^2y^4 + 4752y^6 + 7776x^5 + 15552x^3y^2$$

$$+ 7776xy^4 - 4833x^4 - 6210x^2y^2 - 1377y^4 - 5704x^3 - 4680xy^2$$

$$- 1944x^2 - 648y^2 - 288x - 16 = 0\}.$$

(2) If $q = 0$, then the boundary of $W_0(N)$ is the circle

$$\{x + iy \in \mathbb{C} : |x + iy| = \frac{27}{22}\}.$$

(3) If $0 < q \leq 1/2$ or $13/14 \leq q < 1$, then the boundary of $W_q(N)$ lies on the algebraic curve defined by $K(x, y : q) = 0$ for a polynomial $K(x, y : q)$ of degree 14 defined by (4.6).

(4) If $1/2 < q < 13/14$, then the boundary of $W_q(N)$ consists of two arcs. One part is a circular arc

$$\{x + iy : (x, y) \in \mathbb{R}^2, (x - \frac{2q}{3})^2 + y^2 = (\frac{4}{3})^2, \frac{4}{3} \leq x \leq \frac{-5 + 2q^2 + 2\sqrt{3}\sqrt{1-q^2}}{q}\}.$$

Another part lies on the curve $K(x, y : q) = 0$ and satisfies $x = \Re(x + iy) > (\frac{-5 + 2q^2 + 2\sqrt{3}\sqrt{1-q^2}}{q})/q$.

We prove this theorem in sections 2-4.

2. Another expression of the $q$-numerical range

A fundamental principle to perform the computation of the boundary of the range can be found in [13], [10], [4]. Let $A$ be an $n$-by-$n$ matrix. We introduce a function $h_A$ on the numerical range $W(A)$ by

$$h_A(z) = \max\{t \in \mathbb{R} : (z, t) \in W(A, A^*A)\}.$$  (2.1)
By the convexity of $W(A, A^*A)$ for $n \geq 3$ or the convexity of the region surrounded by $W(A, A^*A)$, this function is concave on $W(A)$. Hence it is continuous in the interior of $W(A)$. By Schwarz's inequality,

$$|\xi^*A\xi|^2 \leq \langle A\xi, A\xi \rangle = \xi^*A^*A\xi$$

and hence $h(z) - |z|^2 \geq 0$. As the sum of two concave functions, the function $z \mapsto h(z) - |z|^2$ is a non-negative concave function. It follows from the concaveness of the function $s \in [0, \infty) \mapsto \sqrt{s}$ that the relation

$$(h(tz_1 + (1-t)z_2) - |tz_1 + (1-t)z_2|^2)^{1/2}$$

$$(\geq (t(h(z_1) - |z_1|^2) + (1-t)(h(z_2) - |z_2|^2))^{1/2}$$

$$(\geq t(h(z_1) - |z_1|^2)^{1/2} + (1-t)(h(z_2) - |z_2|^2))^{1/2}$$

holds for $z_1, z_2 \in W(A), 0 \leq t \leq 1$. Hence the function $(h(z) - |z|^2)^{1/2}$ on $W(A)$ is also concave, and the formula (1.2) is rewritten as

$$W_q(A) = \{qz + \sqrt{1 - q^2}w \sqrt{h(z)} - |z|^2 : z \in W(A), w \in C, |w| \leq 1\}.$$  (2.2)

We introduce a compact convex set $\Gamma(A)$ by

$$\Gamma(A) = \{(x_1, x_2, u_1, u_2) \in \mathbb{R}^4 : x_1 + ix_2 \in W(A), x_1^2 + x_2^2 + u_1^2 + u_2^2 \leq h(x_1 + ix_2)\}.$$  (2.3)

Define an orthogonal projection $\Pi_q$ of $\mathbb{R}^4$ onto $C \cong \mathbb{R}^2$ by

$$\Pi_q(x_1, x_2, u_1, u_2) = (qx_1 + \sqrt{1 - q^2}u_1) + i(qx_2 + \sqrt{1 - q^2}u_2).$$  (2.4)

Then the formula (2.2) becomes

$$W_q(A) = \Pi_q(\Gamma(A)).$$  (2.5)

3. Boundary of the Davis-Wielandt shell

We shall determine the equation of the boundary of the Davis-Wielandt shell $W(N, N^*N)$. We use the equation

$$\{ux + vy + wz : (x, y, z) \in \mathbb{R}^3, (x + iy, z) \in W(N, N^*N)\}$$

$$= [\lambda_3(u\Re(N) + v\Im(N) + wN^*N), \lambda_1(u\Re(N) + v\Im(N) + wN^*N)],$$  (3.1)
for every non-zero vector \((u, v, w) \in \mathbb{R}^3\), where
\[
\lambda_1(H) \geq \lambda_2(H) \geq \lambda_3(H)
\]
are eigenvalues of a \(3 \times 3\) Hermitian matrix \(H\). By this equation, every boundary point \((x + iy, z)\) of \(W(N, N^* N)\) lies on the dual surface \(DS\) of the surface \(F_N(X, Y, Z, 1) = 0\) or a tangent plane of the surface \(DS\). By the elimination method, we compute that the dual surface \(DS\) is defined by a quartic polynomial
\[
L(x, y, z) = 272x^4 + 528x^2y^2 + 256y^4 + 168x^3z + 168xy^2z + 249x^2z^2 + 249y^2z^2 + 132z^4
\]
\[
+ 32x^3 + 32xy^2 - 360x^2z - 344y^2z - 168xz^2 + 16x^2 + 16y^2
\]
\[
- 32x + 88z^2 - 16z.
\]
(3.2)
The dual surface \(L(x, y, z) = 0\) has a tangent plane
\[
L_0 = 3z - 4x - 4 = 0.
\]
(3.3)
This plane and the quartic surface meet at the ellipse \(E_0:\)
\[
\frac{49}{4} \left( x + \frac{2}{7} \right)^2 + \frac{21}{4} y^2 = 1,
\]
(3.4)
on the above plane. On the interior of the convex domain \(W(N)\), the function \(\psi(x, y) = \sqrt{z - (x^2 + y^2)}\) is continuously differentiable. On the closed domain \(E\) bounded by the above ellipse, the function \(\psi(x, y) = \sqrt{z - (x^2 + y^2)}\) is represented by
\[
\psi(x, y) = \frac{4}{3} \left[ 1 - \frac{9}{16} (x - \frac{2}{3})^2 - \frac{9}{16} y^2 \right]^{1/2},
\]
(3.5)
and its graph lies on an ellipsoid. For \(0 < q < 1\), we consider the set
\[
\Delta_q(N) = \{ x + iy : (x, y) \in \mathbb{R}^2, (\frac{d\psi}{dx})^2 + (\frac{d\psi}{dy})^2(x, y) = \frac{q^2}{1 - q^2} \}.
\]
(3.6)
Then the boundary of \(W_q(N)\) satisfies
\[
\partial W_q(N) \subseteq \{ q(x + iy) + \sqrt{1 - q^2 w\psi(x, y)} : x + iy \in \Delta_q(N), w \in \mathbb{C}, |w| = 1 \}.
\]
(3.7)
(cf. [4], Theorem 2). The intersection of \(\Delta_q(N)\) with the elliptical disc \(E:\)
\[
\frac{49}{4} \left( x + \frac{2}{7} \right)^2 + \frac{21}{4} y^2 \leq 1,
\]
(3.8)
is non-empty if and only if $1/2 \leq q \leq 13/14$. In the case $q = 1/2$ or $q = 13/14$, this set is a singleton on the ellipse $(49/4)(x + \frac{3}{2})^2 + (21/4)y^2 = 1$. In the case $1/2 < q < 13/14$, the intersection coincides with the intersection of the circle

$$\{x + iy : (x, y) \in \mathbb{R}^2, (x - \frac{2}{3})^2 + y^2 = (\frac{4q}{3})^2\}$$

and the elliptical disc $(49/4)(x + \frac{2}{3})^2 + (21/4)y^2 \leq 1$. The function $\psi$ on $W(N)$ attains its maximum $27/22$ at $(x, y) = (3/11, 0)$. Corresponding to this fact, the range $W_0(N)$ is the circular disc $|z| \leq 27/22$. The boundary of $W_1(N)$ lies on the sextic curve in Theorem 1.1 (1).

4. Proof of Theorem 1.1

We shall prove Theorem 1.1 for $0 < q < 1$, that is (3), (4) of Theorem 1.1. By (2.5), every point $x + iy$ of $\partial W_q(N)$ is expressed as

$$x = qX_0 + \sqrt{1 - q^2}U, \quad y = qY_0 + \sqrt{1 - q^2}V; \quad (4.1)$$

for some boundary point $(X_0, Y_0, U, V)$ of $\Gamma(N)$. If $0 < q \leq 1/2$ or $13/14 \leq q < 1$, then the above $(X_0, Y_0, U, V)$ is chosen so that

$$L_0(X, Y, U, V) = L(X, Y, X^2 + Y^2 + U^2 + V^2) = 0, \quad (4.2)$$

and

$$L_1(X, Y : x, y, q) = L_0(X, Y, \frac{1}{\sqrt{1 - q^2}}(x - qX), \frac{1}{\sqrt{1 - q^2}}(y - qY)), \quad (4.3)$$

satisfies

$$\frac{\partial L_1}{\partial X}(X_0, Y_0 : x, y, q) = \frac{\partial L_1}{\partial Y}(X_0, Y_0 : x, y, q) = 0. \quad (4.4)$$

By the relation (3.7), the boundary of $W_q(N)$ is also viewed as the envelop of the 1-parameter family of circles. In the case $1/2 < q < 13/14$, the set $\Delta_q(N) \cap E$ is a circular arc. But the whole $\Delta_q(N)$ is not contained in $E$. One
part of the boundary of $W_q(N)$ satisfies the above equations. Another part of $\partial W_q(N)$ lies on the outer envelope of the 1-parameter family of the circles:

\[
\{ x + iy : (x, y) \in \mathbb{R}^2, (X, Y) \in \mathbb{R}^2, (x - qX)^2 + (y - qY)^2 = (1 - q^2) \frac{4 + 4X - 3X^2 - 3Y^2}{3}, \]

\[
(X - \frac{2}{3})^2 + Y^2 = \left( \frac{4q}{3} \right)^2 \left( \frac{49}{4} (X + \frac{2}{7})^2 + \frac{21}{4} Y^2 \leq 1 \right).
\]

and hence it lies on the circle

\[
(x - \frac{2q}{3})^2 + y^2 = \left( \frac{4}{3} \right)^2, \quad (4.5)
\]

The main computations to obtain the boundary of the equation of $W_q(N)$ (0 < $q$ < 1) consist in the eliminations of $X_0, Y_0$ from the equations (4.3), (4.4). We perform the successive eliminations of $Y_0$ and $X_0$. The equation of the envelope of the family of circles is obtained as a simple factor of the resultant of $K_0(x, y : X, q)$ and $\partial K_0(x, y : X, q) / \partial X$ with respect to $X$, where $K_0(x, y : X, q)$ is a simple factor of the resultant of $L_1(X, Y : x, y)$ and $\partial L_1(X, Y : x, y, q) / \partial Y$ with respect to $Y$. By direct long computations by a computer, we obtain the equation of the boundary of $W_q(N)$

\[
K(x, y : q) = 203233536 (x^2 + y^2)^5 \{(11q^2 + 16)x^4 + 2(297q^4 - 256q^2 + 256)x^2y^2 + (27q^2 - 16)y^4 \}
\]

\[
+ 98537472q x (x^2 + y^2) \{(-605q^4 + 616q^2 + 2176)x^4 + (-2040q^4 + 664q^2 + 4352)x^2y^2
\]

\[
+(-1377q^4 + 48q^2 + 2176)y^4 + 23328 (x^2 + y^2)^4 \{3147089q^8 - 5573106q^4 + 6132288q^2 - 4350464 \}
\]

\[
+(8544690q^6 - 31872868q^4 + 32253440q^2 - 8700928)x^2y^2 + (8109153q^6 + 30362994q^4 + 26121152q^2
\]

\[
- 4350464) y^4 + 15552q x (x^2 + y^2)^3 \{(-4146199q^6 + 13793770q^4 + 5518656q^2 - 28651008)x^4
\]

\[
+(-12162138q^6 + 25626092q^4 + 27657600q^2 - 57302016)x^2y^2 + (-6619293q^6 + 9033450q^4
\]

\[
+ 22138944y^2 - 28651008)y^4 + 81(x^2 + y^2)^2 \{(523461745q^8 - 212487788q^6 + 6290634120q^4
\]

\[
- 7032029184q^2 + 429981696)x^6 + (1782657363q^6 - 9976494272q^2 + 24892826112q^4
\]

\[
-21265956864q^6 + 1289945088)x^4 y^2 + (2777208147q^8 - 14892417216q^6 + 30795194880q^4
\]

\[
-21455826176q^8 + 1289945088)x^4 y^2 + (14949438579q^8 - 6994663488q^6 + 12193003008q^4
\]

\[
-7201898496q^8 + 429981696y^8 + 108q x (x^2 + y^2)^2 \{(-232777441q^8 + 851960576q^6
\]

\[
+ 1001346560q^6 - 3116924928q^2 + 1289945088)x^4 + (-797967810q^8 + 1586611712q^4
\]

\[
+ 281905868q^8 - 6506348544q^2 + 2579890176)x^2 y^2 + (-433553121q^8 + 631431936q^6 + 1816598016q^4
\]

\[
- 3389423616q^2 + 1289945088)y^4 + 36q^2 (x^2 + y^2) \{(432453157q^8 - 2256714240q^6 + 8021457408q^4
\]
some function. Figure 2 is produced by using the formula (2.2).

Thus we proved Theorem 1.1.

By this way, we determine the changing points $(x_1, \pm y_1)$ of the circular arc of $\partial W_q(N)$ and another part of $\partial W_q(N)$. It satisfies

$$9q^2 x_1^2 + (-12q^3 + 30q) x_1 + (4q^4 - 8q^2 + 13) = 0,$$

and $x_1 \geq (2q/3) - 4/3$, hence

$$x_1 = \frac{-5 + 2q^2 + 2\sqrt{3(1-q^2)}}{q}.$$

Thus we proved Theorem 1.1.

We present two graphics. Figure 1 is produced by plotting the contour of some function. Figure 2 is produced by using the formula (2.2).
Figure 1: the graphic of the curve $K(x, y : 13/14) = 0$.

Figure 2: the numerical aproximation of the range $W_{13/14}(N)$ of $N$ given by (1.5) by using C. K. Li’s computer program ([11]).

References


[11] C. K. Li, the $q$-numerical range of a general matrix, Matlab program presented in his home page.


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