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On a Class of Meromorphic Multivalent Functions with Negative Coefficients Defined by Ruscheweyh Derivative

S. M. Khairnar and Meena More

Department of Mathematics
Maharashtra Academy of Engineering
Alandi - 412 105, Pune (M. S.), India
smkhairnar2007@gmail.com, meenamore@maepune.com

Abstract

In this paper we have introduced a subclass $H(\lambda, p, \ell, A, B, \alpha)$ of meromorphic multivalent functions with negative coefficients in the unit punctured disc. We derive basic properties like coefficient inequality, distortion theorem, radius of starlikeness and convexity, Hadamard product, extreme points, closure theorems for functions belonging to this class.

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1. Introduction

Let M denote the class of function of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \tag{1.1}$$

$p \in I\!N$, which are meromorphic multivalent in the punctured unit disc $U^* = \{z : 0 < z < 1\}$.

Consider a subclass T of functions of the form

$$f(z) = \frac{1}{z^p} - \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \tag{1.2}$$

$p \in I\!N, a_{p+k} \geq 0$.

We aim to study the class $H(\lambda, p, \ell, A, B, \alpha)$ consisting of functions $f \in T$ and satisfying

$$\left| \frac{z^{p+1}[(1-\lambda p)(D^{n+p-1}f)' + \lambda p(D^{n+p-1}f)'] + p}{Bz^{p+1}(D^{n+p-1}f)' + (A-B)p + p - \ell} \right| < \alpha$$

$0 \leq \lambda \leq 1, p \in I\!N, 0 \leq \ell < p, -1 \leq A < B \leq 1, 0 < \alpha \leq 1.$

$D^{n+p-1}f$ is the $(n+p-1)$-th order Ruscheweyh derivative defined by

$$\begin{aligned} D^{n+p-1}f(z) &= \frac{z^{-p}}{(1-z)^{n+p}} * f(z) \\ &= \frac{1}{z^p} - \sum_{k=0}^{\infty} B_{k+2p-1}(n,p) a_{p+k} z^{p+k} \end{aligned}$$

where $B_{k+2p-1}(n,p) = \begin{pmatrix} n+3p+k-1 \\ 2p+k \end{pmatrix}$.

2. Coefficient Inequality

Theorem 1 : Let f be defined by (1.2). Then $f \in H(\lambda, p, \ell, A, B, \alpha)$ if and only if

$$\sum_{k=0}^{\infty} (p+k)(1-B\alpha) B_{k+2p-1}(n,p) a_{p+k} \leq \alpha(p(2B-A) - (p-\ell)). \tag{2.1}$$

Proof : Assume (2.1) holds. We show that $f \in H(\lambda, p, \ell, A, B, \alpha)$. Consider

$$\begin{aligned} &|z^{p+1}[(1-\lambda p)(D^{n+p-1}f)' + \lambda p(D^{n+p-1}f)'] + p| - \alpha|Bz^{p+1}(D^{n+p-1})' \\ &+(A-B)p + p - \ell| \\ &= \left| \sum_{k=0}^{\infty} (p+k) B_{k+2p-1}(n,p) a_{p+k} z^{2p+k} \right| \\ &-\alpha \left| \sum_{k=0}^{\infty} B(p+k) B_{k+2p-1}(n,p) a_{p+k} z^{2p+k} - (p(A-2B) + p - \ell) \right| \qquad (2.2) \\ &\leq \sum_{k=0}^{\infty} (p+k)(1-B\alpha) B_{k+2p-1}(n,p) a_{p+k} + \alpha(p(A-2B) + p - \ell) \leq 0. \end{aligned}$$

by using maximum modulus principle and (2.1).

Hence $f \in H(\lambda, p, \ell, A, B, \alpha)$.

Conversely, suppose that $f \in H(\lambda, p, \ell, A, B, \alpha)$ thus

$$\left| \frac{z^{p+1}[(1-\lambda p)(D^{n+p-1}f)' + \lambda p(D^{n+p-1}f)'] + p}{Bz^{p+1}(D^{n+p-1}f)' + (A-B)p + p - \ell} \right| < \alpha$$

that is

$$\left|\frac{\sum\limits_{k=0}^{\infty}(p+k)B_{k+2p-1}(n,p)a_{p+k}z^{2p+k}}{\sum\limits_{k=0}^{\infty}B(p+k)B_{k+2p-1}(n,p)a_{p+k}z^{2p+k}-(p(A-2B)+p-\ell)}\right|<\alpha.$$

Using the fact that $|Re(z)|\leq|z|$ for any z

$$Re\left\{\left|\frac{\sum\limits_{k=0}^{\infty}(p+k)B_{k+2p-1}(n,p)a_{p+k}z^{2p+k}}{\sum\limits_{k=0}^{\infty}B(p+k)B_{k+2p-1}(n,p)a_{p+k}z^{2p+k}-(p(A-2B)+p-\ell)}\right|\right\}<\alpha. \tag{2.3}$$

Now choosing values of z on real axis so that $(D^{n+p-1}f)'$ is real. After clearing denominator in (2.3) and allowing $z\to 1^-$ through real values, we notice that (2.3) will hold provided

$$\sum_{k=0}^{\infty}(p+k)(1-B\alpha)B_{k+2p-1}(n,p)a_{p+k}\leq\alpha(p(2B-A)-(p-\ell)).$$

Hence the result follows.

Corollary 1 : If the function $f(z)\in H(\lambda,p,\ell,A,B,\alpha)$ then

$$a_{p+k}\leq\frac{\alpha(p(2B-A)-(p-\ell))}{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}$$

for $k=0,1,2,\cdots$ with equality for the function

$$f(z)=\frac{1}{z^p}-\frac{\alpha(p(2B-A)-(p-\ell))}{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}z^{p+k},\quad k=0,1,2,\cdots.$$

3. Distortion and Covering Theorem

Theorem 2 : If the function $f\in H(\lambda,p,\ell,A,B,\alpha)$ then

$$\frac{1}{|z|^p}-\frac{\alpha(p(2B-A))-(p-\ell))}{p(1-B\alpha)B_{2p-1}(n,p)}|z|^p\leq|f(z)|\leq\frac{1}{|z|^p}+\frac{\alpha(p(2B-A)-(p-\ell))}{p(1-B\alpha)B_{2p-1}(n,p)}|z|^p.$$

The result is sharp and attained for

$$f(z)=\frac{1}{z^p}-\frac{\alpha(p(2B-A)-(p-\ell))}{p(1-B\alpha)B_{2p-1}(n,p)}z^p$$

at $z=r, r=e^{i\frac{\pi}{2p}}$.

Proof :

$$\begin{aligned} |f(z)| &= \left| \frac{1}{z^p} - \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \right| \\ &\leq \frac{1}{|z|^p} + \sum_{k=0}^{\infty} a_{p+k} |z|^{p+k} \\ &\leq \frac{1}{|z|^p} + |z|^p \sum_{k=0}^{\infty} a_{p+k}. \end{aligned}$$

By Theorem 1

$$\sum_{k=0}^{\infty} a_{p+k} \leq \frac{\alpha(p(2B-A)-(p-\ell))}{p(1-B\alpha)B_{2p-1}(n,p)}. \tag{3.1}$$

Thus

$$|f(z)| \leq \frac{1}{|z|^p} + \frac{\alpha(p(2B-A)-(p-\ell))}{p(1-B\alpha)B_{2p-1}(n,p)}|z|^p$$

$$\begin{aligned} |f(z)| &\geq \frac{1}{|z|^p} - \sum_{n=0}^{\infty} a_{p+k} |z|^{p+k} \\ &\geq \frac{1}{|z|^p} - |z|^p \sum_{k=0}^{\infty} a_{p+k} \\ &\geq \frac{1}{|z|^p} - \frac{\alpha(p(2B-A)-(p-\ell))}{p(1-B\alpha)B_{2p-1}(n,p)}|z|^p. \end{aligned}$$

Theorem 3 : If $f \in H(\lambda, p, \ell, A, B, \alpha)$, then

$$\begin{aligned} &\frac{p}{|z|^{p+1}} - \frac{\alpha(p(2B-A)-(p-\ell))}{(1-B\alpha)B_{2p-1}(n,p)}|z|^{p-1} \leq |f'(z)| \\ &\leq \frac{p}{|z|^{p+1}} \frac{\alpha(p(2B-A)-(p-\ell))}{p(1-B\alpha)B_{2p-1}(n,p)}|z|^{p-1} \end{aligned}$$

with equality for

$$f(z) = \frac{1}{z^p} - \frac{\alpha(p(2B-A)-(p-\ell))}{p(1-B\alpha)B_{2p-1}(n,p)}z^p$$

at $z = r, re^{i\frac{\pi}{2p}}$.

Proof : Notice that

$$\begin{aligned} &(1-B\alpha)B_{2p-1}(n,p) \sum_{k=0}^{\infty} (p+k) a_{p+k} \\ &\leq \sum_{k=0}^{\infty} (p+k)(1-B\alpha)B_{k+2p-1}(n,p) a_{p+k} \leq \alpha(p(2B-A)-(p-\ell)) \end{aligned} \tag{3.2}$$

from Theorem 1. Thus

$$\begin{aligned} |f'(z)| &\le \frac{p}{|z|^{p+1}} + \sum_{k=0}^{\infty}(p+k)a_{p+k}|z|^{p+k-1} \\ &\le \frac{p}{|z|^{p+1}} + |z|^{p-1}\sum_{k=0}^{\infty}(p+k)a_{p+k} \\ &\le \frac{p}{|z|^{p+1}} + \frac{\alpha(p(2B-A)-(p-\ell))}{(1-B\alpha)B_{2p-1}(n,p)}|z|^{p-1}. \end{aligned} \tag{3.3}$$

On the other hand

$$\begin{aligned} |f'(z)| &\ge \frac{p}{|z|^{p+1}} - \sum_{k=0}^{\infty}(p+k)a_{p+k}|z|^{p+k-1} \\ &\ge \frac{p}{|z|^{p+1}} - |z|^{p-1}\sum_{k=0}^{\infty}(p+k)a_{p+k} \\ &\ge \frac{p}{|z|^{p+1}} - \frac{\alpha(p(2B-A)-(p-\ell))}{(1-B\alpha)B_{2p-1}(n,p)}|z|^{p-1}. \end{aligned} \tag{3.4}$$

Combining (3.2) and (3.3) we get the result.

Theorem 4 : The disc $|z| < 1$ is mapped onto a domain that contains the disc

$$|w| < \frac{p(1-B\alpha)B_{2p-1}(n,p) - \alpha p(2B-A) + \alpha(p-\ell)}{p(1-B\alpha)B_{2p-1}(n,p)}.$$

The result is sharp with extremal function

$$\frac{1}{z^p} - \frac{\alpha(p(2B-A)-(p-\ell))}{p(1-B\alpha)B_{2p-1}(n,p)}z^p.$$

Proof : The result follows upon allowing $|z| \to 1$ in Theorem 2.

4. Radius of Convexity

Theorem 5 : Let $f \in H(\lambda, p, \ell, A, B, \alpha)$ then f is p-valently convex in $0 < |z| < R$ of order $\delta, 0 \le \delta < p$, where

$$R = \inf_k \left\{ \frac{p(p-\delta)(1-B\alpha)B_{k+2p-1}(n,p)}{(p+k+\delta)\alpha(p(2B-A)-(p-\ell))} \right\}^{\frac{1}{2p+k}}.$$

Proof : f is p-valently convex of order $\delta, 0 \le \delta < p$ if

$$-Re\left(1 + z\frac{f''}{f'}\right) > \delta.$$

Thus it is enough to show that

$$\left|z\frac{f''}{f'}+1+p+\delta\right| \leq \left|z\frac{f''}{f'}+1-p+\delta\right|.$$

Now

$$\left|\frac{z\frac{f''}{f'}+1+p+\delta}{z\frac{f''}{f'}+1-p+\delta}\right| = \left|\frac{-p\delta-\sum\limits_{k=0}^{\infty}(p+k)(2p+k+\delta)a_{p+k}z^{2p+k}}{p(2p-\delta)-\sum\limits_{k=0}^{\infty}(p+k)(k+\delta)a_{p+k}z^{2p+k}}\right|$$

is bounded above by 1 if

$$2p(\delta-p)+\sum_{k=0}^{\infty}(p+k)(2p+2k+2\delta)a_{p+k}|z|^{2p+k} \leq 1.$$

That is if

$$|z|^{2p+k} \leq \frac{p(p-\delta)(1-B\alpha)B_{k+2p-1}(n,p)}{(p+k+\delta)\alpha(p(2B-A)-(p-\ell))}.$$

Hence

$$R=\inf_k\left\{\frac{p(p-\delta)(1-B\alpha)B_{k+2p-1}(n,p)}{(p+k+\delta)\alpha(p(2B-A)-(p-\ell))}\right\}^{\frac{1}{2p+k}}.$$

The estimate is sharp for

$$f(z)=\frac{1}{z^p}-\frac{\alpha(p(2B-A)-(p-\ell))}{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}z^{p+k}$$

for some k.

5. Extreme Points

Theorem 6 : Let $f_{p-1}(z)=\frac{1}{z^p}$ and

$$f_{p+k}=\frac{1}{z^p}-\frac{\alpha(p(2B-A)-(p-\ell))}{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}z^{p+k}$$

for $k=0,1,2,\cdots$. Then $f\in H(\lambda,p,\ell,A,B,\alpha)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=-1}^{\infty}\mu_{p+k}f_{p+k}(z)$$

where $\mu_{p+k}\geq 0$ and $\sum\limits_{k=-1}^{\infty}\mu_{p+k}=1$.

Proof : Suppose that $f(z)=\sum\limits_{k=-1}^{\infty}\mu_{p+k}f_{p+k}(z)$

$$f(z)=\frac{1}{z^p}-\sum_{k=0}^{\infty}\frac{\alpha(p(2B-A)-(p-\ell))}{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}\mu_{p+k}z^{p+k}.$$

Now $f\in H(\lambda,p,\ell,A,B,\alpha)$ since

$$\sum_{k=0}^{\infty}\frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))}\times\frac{\alpha(p(2-A)-(p-\ell))}{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}\mu_{p+k}$$
$$=\sum_{k=0}^{\infty}\mu_{p+k}\le 1-\mu_{p-1}\le 1.$$

Conversely, assume $f\in H(\lambda,p,\ell,A,B,\alpha)$. Then we show that f can be written in the form

$$\sum_{k=-1}^{\infty}\mu_{p+k}f_{p+k}(z).$$

Now $f\in H(\lambda,p,\ell,A,B,\alpha)$ implies from Theorem 1

$$a_{p+k}\le\frac{\alpha(p(2B-A)-(p-\ell))}{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}$$

setting

$$\mu_{p+k}=\frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))}a_{p+k},\quad k=0,1,2,\cdots$$

and $\mu_{p-1}=1-\sum\limits_{k=0}^{\infty}\mu_{p+k}$ we obtain

$$f(z)=\sum_{k=-1}^{\infty}\mu_{p+k}f_{p+k}(z).$$

6. Hadamard Product

Theorem 7 : Let $f,g\in H(\lambda,p,\ell,A,B,\alpha)$ then $f*g\in H_n(\lambda,p,\ell,A,B,\gamma)$ for

$$f=z-\sum_{k=0}^{\infty}a_{p+k}z^{p+k}$$

$$g=z-\sum_{k=0}^{\infty}b_{p+k}z^{p+k}\quad\text{and}$$

$$f * g = z - \sum_{k=0}^{\infty} a_{p+k} b_{p+k} z^{p+k}$$

where

$$\gamma \geq \frac{\alpha^2(p(2B-A)-(p-\ell))}{(p+k)(1-B\alpha)^2 B_{k+2p-1}(n,p) + B\alpha^2(p(2B-A)-(p-\ell))}.$$

Proof : $f \in H(\lambda, p, \ell, A, B, \alpha)$ and so

$$\sum_{k=0}^{\infty} \frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} \quad a_{p+k} \leq 1 \tag{5.1}$$

$$\sum_{k=0}^{\infty} \frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} \quad b_{p+k} \leq 1. \tag{5.2}$$

We have to find the smallest number γ such that

$$\sum_{k=0}^{\infty} \frac{(p+k)(1-B\gamma)B_{k+2p-1}(n,p)}{\gamma(p(2B-A)-(p-\ell))} \quad a_{p+k} b_{p+k} \leq 1. \tag{5.3}$$

By Cauchy Schwarz inequality

$$\sum_{k=0}^{\infty} \frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} \quad \sqrt{a_{p+k} b_{p+k}} \leq 1. \tag{5.4}$$

Therefore it is enough to show that

$$\frac{(p+k)(1-B\gamma)B_{k+2p-1}(n,p)}{\gamma(p(2B-A)-(p-\ell))} \quad a_{p+k} b_{p+k}$$

$$\leq \frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} \quad \sqrt{a_{p+k} b_{p+k}}.$$

That is

$$\sqrt{a_{p+k} b_{p+k}} \leq \frac{\gamma(1-B\alpha)}{\alpha(1-B\gamma)}. \tag{5.5}$$

From (5.4)

$$\sqrt{a_{p+k} b_{p+k}} \leq \frac{\alpha(p(2B-A)-(p-\ell))}{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}.$$

Thus it is enough to show that

$$\frac{\alpha(p(2B-A)-(p-\ell))}{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)} \leq \frac{\gamma(1-B\alpha)}{\alpha(1-B\gamma)}$$

which simplifies to

$$\gamma \geq \frac{\alpha^2(p(2B-A)-(p-\ell))}{(p+k)(1-B\alpha)^2 B_{k+2p-1}(n,p) + B\alpha^2(p(2B-A)-(p-\ell))}.$$

7. Closure Theorem

Theorem 8 : Let $f_j \in H(\lambda, p, \ell, A, B, \alpha), j = 1, 2, \cdots, s$ then

$$g(z) = \sum_{j=1}^{s} c_j f_j(z) \in H(\lambda, p, \ell, A, B, \alpha)$$

for $f_j(z) = \frac{1}{z^p} - \sum\limits_{k=0}^{\infty} a_{p+k,j} z^{p+k}$ where $\sum\limits_{j=1}^{s} c_j = 1$.

Proof :

$$\begin{aligned} g(z) &= \sum_{j=1}^{s} c_j f_j(z) \\ &= \frac{1}{z^p} - \sum_{k=0}^{\infty} \sum_{j=1}^{s} c_j a_{p+k,j} z^{p+k} \\ &= \frac{1}{z^p} - \sum_{k=0}^{\infty} e_k z^{p+k} \quad \text{where} \quad e_k = \sum_{j=1}^{s} c_j a_{p+k,j}. \end{aligned}$$

Thus $g(z) \in H(\lambda, p, \ell, A, B, \alpha)$ if

$$\sum_{k=0}^{\infty} \frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} e_k \le 1$$

that is if

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_{j=1}^{s} \frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} c_j a_{p+k,j} \\ &\sum_{j=1}^{s} c_j \sum_{k=0}^{\infty} \frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} a_{p+k,j} \\ &\le \sum_{j=1}^{s} c_j \le 1 \quad \text{since} \quad \frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} a_{p+k,j} \le 1 \end{aligned}$$

as $f_j(z) \in H(\lambda, p, \ell, A, B, \alpha)$

Theorem 9 : Let $f, g \in H(\lambda, p, \ell, A, B, \alpha)$ then

$$h(z) = z - \sum_{k=0}^{\infty} (a_{p+k}^2 + b_{p+k}^2) z^{p+k}$$

belongs to $H(\lambda, p, \ell, A, B, \gamma)$ where

$$\gamma \ge \frac{2\alpha^2(p(2B-A)-(p-\ell))}{(p+k)(1-B\alpha)^2 B_{k+2p-1}(n,p) + 2B\alpha^2(p(2B-A)-(p-\ell))}.$$

Proof : f and $g \in H(\lambda, p, \ell, A, B, \alpha)$ and so

$$\sum_{k=0}^{\infty} \left[\frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} \right]^2 a_{p+k}^2$$

$$\leq \left(\sum_{k=0}^{\infty} \frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} \right)^2 \leq 1. \tag{7.1}$$

Similarly

$$\sum_{k=0}^{\infty} \left[\frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} \right]^2 b_{k+p}^2 \leq 1. \tag{7.2}$$

We show that

$$\sum_{k=0}^{\infty} \frac{(p+k)(1-B\gamma)_k(n,p)}{\gamma(p(2B-A)-(p-\ell))} (a_{p+k}^2 + b_{p+k}^2) \leq 1$$

that is $h \in H(\lambda, p, \ell, A, B, \gamma)$.

Adding (7.1) and (7.2)

$$\sum_{k=0}^{\infty} \frac{1}{2} \left[\frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} \right]^2 (a_{p+k}^2 + b_{p+k}^2) \leq 1.$$

Thus it is enough to show that

$$\frac{(p+k)(1-B\gamma)B_{k+2p-1}(n,p)}{\gamma(p(2B-A)-(p-\ell))} \leq \frac{1}{2} \left[\frac{(p+k)(1-B\alpha)B_{k+2p-1}(n,p)}{\alpha(p(2B-A)-(p-\ell))} \right]^2.$$

Simplifying we get

$$\gamma \geq \frac{2\alpha^2(p(2B-A)-(p-\ell))}{(p+k)(1-B\alpha)^2 B_{k+2p-1}(n,p) + 2B\alpha^2(p(2B-A)-(p-\ell))}.$$

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