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# A Method for Solving Multilevel Multi-Objective System Based on Linguistic Judgment Matrix[1]

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**Abstract**

In this paper, based on theory of linguistic judgment matrix and that of feedback system, we present a new method for solving Multilevel Multi-objective System problem. It is the generalization of weighted method, and has more extensive application in practice.

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**Keywords:** linguistic judgment matrix; multilevel multi-objective system; interactive programming

## 1 Introduction

The multilevel multi-objective system has extensive existences in management fields. Usually, this kind of problem can be solved by multiple mathematics programming. Most of studies in this field are focused on bilevel programming[1][2][3]. However, many practical problems should be modeled as multilevel multi-objective program, and we are required to develop a new appropriate methods. The multilevel multi-objective program considered in this paper can be formulated as follows

$$\min_{x^k} \ f_0^k(x,y) \qquad k=1,2,\ldots,n_0.$$

$$s.t. \quad (x,y)\in\Omega_0. \tag{1.1}$$

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$$\min_{y_j^i} \ f_j^i(x,y)$$

$$s.t. \ \ (x,y) \in \Omega_j \ \ \ j = 0,1,2,\dots,p.$$

where $x = (x^1, x^2, \dots, x^{n_0})$, $y = (y^1, y^2, \dots, y^p)$ are controlled by the highest and the i-th level, respectively. $\Omega_0, f_0^k$ indicate the decision constraint set and the k-th objective function of the highest layer, respectively. $y_j^i, \Omega_j, f_j^i$ indicate the decision variable, constraint set and the j-th objective function of the i-th layer, respectively. The multilevel multi-objective program has $p+1$ layers. There are $n_i$ decision makers every layerand each decision maker has decision variable and objective function of himself. The hierarchical nature of the problem is reflected by the order imposed on the choice of the decision. One level makes his decision according to that of his higher level. It is usually assumed that the leader masters the information of the followers' objectives and constraints, while the followers make their decisions after the leader's strategy is announced. Lower decisions also have effect on upper objective function, while upper decision makers may adjust their decision until their objective functions were satisfied. Decision makers of the same layer have common constraints and make decision cooperatively.

Since solving multiple level multi-objective programming is very complex, it is seldom that there exists globally optimal solution. In this paper, based on theory of linguistic judgment matrix and that of feedback system, we present a new method for solving multilevel multi-objective System problem. It is the generalization of weighted method, and has more extensive application in practice.

# 2 Principle and Method

The rational pairwise comparison matrix $P = (p_{ij})_{n\times n}$ is provided by the decision makerwhere $p_{ij}$ is selected from the linguist label set $S = \{s_0, s_1, \dots, s_{2g}\}$, which denotes their importance ratio of the objective function $f_i$ and $f_j$. Especially, $p_{ij} = s_g$ indicates indifference between $f_i$ and $f_j$ of the same level. The linguist label set $S$ includes $2G+1$ elements($G$ is integer). We give the following two definition [4].

**Definition 2.1** *Let $S = \{s_0, s_1, \dots, s_{2g}\}$ be a finite and totally ordered discrete linguistic label setif $s_i \in S$ is i-th elementsthen its subscript can be obtained from the function as follows*

$$I : S \to N$$
$$I(s_i) = i, \quad s_i \in S$$

*where $N$ is the set of integer*

**Definition 2.2** *$Q$ is the derived matrix of the linguistic judgment matrix $P$, if $Q = (q_{ij})_{n\times n}$, where $q_{ij} = e^{I(p_{ij})-g}$, $p_{ij} \in S$, $i, j = 1, 2, \dots, n$.*

**Theorem 2.1** *The derived matrix $Q$ of the linguistic judgment matrix $P$ is reciprocal matrix.*

*Proof.* By the composing of $P$, we know

$$I(p_{ii}) = g, I(p_{ij}) + I(p_{ji}) = 2g, \forall q_{ij}, q_{ji} \in Q,$$

then

$$q_{ij} \cdot q_{ji} = e^{I(p_{ij})-g} \cdot e^{I(p_{ji})-g} = e^{I(p_{ij})+I(p_{ji})-2g} = e^{2g-2g} = e^0 = 1.$$

especially

$$q_{ii} = d^{I(p_i i)-g} = d^{g-g} = 1.$$

So $Q$ is reciprocal matrix. □

From above, we know that the multiple level multi-objective system problem can be transformed into the a feedback system with interior dependence. Based on linguist preference, from Theorem 2.1 and the theory of pairwise compare matrix, we can transform linguist matrix into pairwise compare matrix and process it to resolve the linguist information of the system. According to the structural characteristic of system, we use the method of constructing the super matrix[5] to structure the super matrix $W$ of multiple level multi-objective programming based on linguist preference information as follows:

$$W = \begin{bmatrix} w_{00} & w_{01} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ w_{10} & w_{11} & w_{12} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & w_{21} & w_{22} & w_{23} & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & w_{p-2,p-2} & w_{p-2,p-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & w_{p-1,p-2} & w_{p-1,p-1} & w_{p-1,p} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & w_{p,p-1} & w_{p,p} \end{bmatrix}$$

where subblock $w_{i,j}(i, j = 0, 1, \dots, p)$ is the priority weight that the decision maker of i-th level acts on for the influence of the j-th level in the multiple level multi-objective system.

**Theorem 2.2** *The super matrix of multiple level multi-objective programming based on linguist preference information is prime matrix.*

*Proof.* The objective functions of each level in multiple multi-objective system can be regard as a node, then the digraph $G(W)$, whose adjacent matrix is $W$, is the digraph of system, and the nodal set is composed by all objective functions [7].

In the digraph of systemif there exists a directed edge from node $i$ to node $j$, then the relation of node $i$ and node $j$ is accessible. By the structure of the system, we can know that there exists a directed circuit between the elements of different level. That is to say, for all $i, j$, there must exist a directed circuit pass through them. So node $i$ and $j$ is accessible to each other. In one words, graph $G(W)$ is strongly connected.

From the characters of the system, we can conclude that the elements between the same level is depended. So the trace of super matrix $W$ is positive, i.e., $trace(W) > 0$. So it can be obtained from above that the super matrix of multiple level multi-objective programming based on linguist preference information is a prime matrix. □

By the fact that the column sum of $W$ is one and the theorem about nonnegative irreducible matrix, the following result can be obtained easily [5].

**Theorem 2.3** *The maximum characteristic root of the super matrix $W$ is $\lambda_{max} = 1$, and it is a simple root. There exists positive eigenvector corresponding to $\lambda_{max}$ and $\lambda_i$, where $\lambda_i$ is the other characteristic root and $|\lambda_i| < 1$.*

**Theorem 2.4** *For a prime matrix $A$ and $x \in R^n$, we have $\lim\limits_{k\to\infty} \frac{A^k x}{x^T A^k x} = cv_1$, where $v_1$ is the eigenvector corresponding to the maximum characteristic root of $A$. Especially, if $x = e = (1, 1, \dots, 1)^T$, then $\lim\limits_{k\to\infty} \frac{A^k e}{e^T A^k e} = w$, where $w$ is the normalized eigenvector of the max characteristic root of $A$.*

The normalized right dominant eigenvector of the super matrix $W$ can be regarded as the vector of limit absolute priority from above. In fact, it is as same as the definition of the vector of limit absolute priority [5].

**Definition 2.3** *If $v^0$ is the vector about initial importance degree in system of all elements, we can obtain the limit of the priority weights about accumulative influence among the elements, i.e., $\lim\limits_{k\to\infty} W^k v^0$, then $\lim\limits_{k\to\infty} W^k v^0$ is called limit absolute priority(LAP). If the limit absolute priority and $v^0$ is independent, then the system is called ergodic.*

**Theorem 2.5** *The vector of limit absolute priority $x$ of super matrix $W$ is the right dominant eigenvector and the multiple level multi-objective system is ergodic.*

*Proof.* From Theorem 2.3, $\lambda_{max} = 1$ is the maximum characteristic root of the super matrix $W$ and it is a simple root. We assume $x$ is the normalized eigenvector of $\lambda_{max}$, i.e., $Wx = x$.

For any normalized initial vector $v^0$, from Theorem 2.3, the spectral radius of super matrix $W$ is one, i.e., $\rho(W) = 1$. So $\lim_{k\to\infty} W^k$ exists, denoted as $W^\infty$. Thus $LAP = W^\infty v^0$.

Assume $v = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then by Theorem 2.4 we can obtain the following conclusion,

$$x = \lim_{k\to\infty} \frac{W^k v}{v^T W^k v} = \frac{\lim\limits_{k\to\infty} W^k v}{\lim\limits_{k\to\infty} v^T W^k v} = W^\infty v = W^\infty v^0.$$

which means that the vector of limit absolute priority is the normalized right dominant eigenvector $x$ and $LAP$ is independent on $v^0$, i.e., the system is ergodic.

# 3 Algorithm

If $w_k > 0 (k = 1, 2, \ldots, n)$, then the optimal solution of weight problem is the non-inferior solution of the multiple objective programming [6]. So the optimal solution of single objective programming (3.1), obtained by the weight priority from the normalized right dominant eigenvector of the linguist super matrix, is the non-inferior solution of the multiple level multi-objective programming (1.1). Hence we propose the weight algorithm for solving the multilevel multi-objective system based on linguistic judgment.

**Algorithm**

Step 1: Structure the pairwise comparison matrix $P = (p_{ij})_{n\times n}$ based on the linguist label set.

Step 2: Obtain the derived matrix $Q$ of the linguistic judgment matrix $P$ by Definition 2.1.

Step 3: Structured the linguist super matrix $W$ of multiple level multi-objective programming according to the methods in [5].

Step 4: Calculate the right dominant eigenvector of linguist super matrix $W$:

$$w = (w_0^1, w_0^2, \ldots, w_0^{n_0}, w_1^1, w_1^2, \ldots, w_1^{n_1}, \ldots, w_i^1, w_i^2, \ldots, w_i^{n_i}, \ldots, w_p^1, w_p^2, \ldots, w_p^{n_p})$$

Step 5: Solve the single objective programming as follows

$$\max_{x^1,x^2,\ldots,x^{n_0},y^1,y^2,\ldots,y^p} \quad \sum_{i=0}^{n_i}\sum_{j=1}^{p} w_i^j f_i^j(x^1,x^2,\ldots,x^{n_0},y^1,y^2,\ldots,y^p)$$

$$s.t.(x^1,x^2,\ldots,x^{n_0},y^1,y^2,\ldots,y^p)\in\Omega_j \quad j=0,1,\ldots,p. \tag{3.1}$$

If $(x^{1^*},x^{2^*},\ldots,x^{n_0^*},y^{1^*},y^{2^*},\ldots,y^{p^*})$ is the optimal solution of programming (3.1), then $(x^{1^*},x^{2^*},\ldots,x^{n_0^*},y^{1^*},y^{2^*},\ldots,y^{p^*})$ is the non-inferior solution of the multiple level multi-objective programming (1.1).

# 4 Conclusion

Since the preference information based on linguist is obtained easily, and it reflects the preference of the decision makers, hence the structure of linguist super matrix embodies the importance degree of decision makers of different levels. However, the priority vector reflects the accumulative influence, so the weighted method is reasonable. It is the generalization of weighted method, and has more extensive application in practice.

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