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Fixed Point Theorems for Generalized Hausdorff Metrics

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Abstract

In this paper, we study the generalized Hausdorff metrics, so-called τ^0-metrics, and establish generalized Kannan's fixed point theorems and generalized Chatterjea's fixed point theorems for multivalued maps. Applying our results, we also present some new fixed point theorems for multivalued maps.

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1. Introduction and preliminaries

Throughout this paper we denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real numbers and the set of positive integers, respectively. Let (X, d) be a metric space. We denote by $\mathcal{CB}(X)$ the class of all nonempty closed bounded subsets of X. For each $x \in X$ and $A \subseteq X$, let $d(x, A) = \inf_{y \in A} d(x, y)$, the distance between x and A. For any A, $B \in \mathcal{CB}(X)$, define a function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ by

$$\mathcal{H}(A, B) = \max \left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\},$$

then $\mathcal{H}$ is said to be the *Hausdorff metric* on $\mathcal{CB}(X)$ induced by the metric d on X. A function $p : X \times X \to [0, \infty)$ is said to be a *τ-function* [4,6], introduced and studied by Lin and Du, if the following conditions hold:

(τ1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;

(τ2) If $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ such that $p(x, y_n) \leq M$ for some $M = M(x) > 0$, then $p(x, y) \leq M$;

(τ3) For any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$;

(τ4) For x, y, $z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

It is known that any w-distance [2,4,5] is a τ-function; see [4, Remark 2.1].

Now, we first introduce the concept of τ^0-functions and τ^0-metrics.

Definition 1.1. Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called a τ^0-function (resp. w^0-distance) if it is a τ-function (resp. w-distance) on X with $p(x, x) = 0$ for all $x \in X$.

Example. Let $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$ and $0 < a < b$. Define the function $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \max\{a(y - x), b(x - y)\}.$$

Then p is nonsymmetric and hence p is not a metric. It is easy to see that p is a τ^0-function.

Definition 1.2. Let (X, d) be a metric space and p be a τ^0-function (resp. w^0-distance). For any A, $B \in \mathcal{CB}(X)$, define a function $\mathcal{D}_p : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ by

$$\mathcal{D}_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\},$$

where $\delta_p(A, B) = \sup_{x\in A} p(x, B)$, then $\mathcal{D}_p$ is said to be the *τ^0-metric* (resp. *w^0-metric*) on $\mathcal{CB}(X)$ induced by p.

Clearly, a Hausdorff metric is a τ^0-metric, but the reverse is not true.

Lemma 1.1. [4] Let (X, d) be a metric space and $p : X \times X \to [0, \infty)$ be any function. If p satisfies (τ3) and there exists a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 1.2. Let A be a closed subset of a metric space (X,d) and $p : X \times X \to [0,\infty)$ be any function. Suppose that p satisfies $(\tau 3)$ and there exists $u \in X$ such that $p(u,u)=0$. Then $p(u,A)=0$ *if and only if* $u \in A$.

Proof. If $u \in A$ with $p(u,u)=0$, then $p(u,A)=0$. Conversely, suppose that $p(u,A)=0$ for some $u \in X$ with $p(u,u)=0$. Let $u_n = u$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \sup\{p(u_n,u_m) : m > n\} = 0$ and there exists a sequence $\{x_n\}$ in A such that $p(u,x_n) \to 0$. By $(\tau 3)$, we have $x_n \to u$. Since A is closed, $u \in A$. $\square$

The following theorem is one of the main results in this paper.

Theorem 1.1. Let (X,d) be a metric space and $\mathcal{D}_p$ a τ^0-metric defined as in Def. 1.2 on $\mathcal{CB}(X)$ induced by a τ^0-function p. Then for A, B, $C \in \mathcal{CB}(X)$, the following hold:

(i) $\delta_p(A,B)=0 \Longleftrightarrow A \subseteq B$;

(ii) $\delta_p(A,B) \le \delta_p(A,C)+\delta_p(C,B)$;

(iii) Every τ^0-*metric* $\mathcal{D}_p$ is a metric on $\mathcal{CB}(X)$.

Proof. We first prove (i). If $\delta_p(A,B)=0$ then $p(a,B)=0$ for all $a \in A$. By Lemma 1.2, we have $A \subseteq B$. Conversely, if $A \subseteq B$, then $\delta_p(A,B)=0$ by Lemma 1.2 again. Therefore $\delta_p(A,B)=0 \Longleftrightarrow A \subseteq B$. The proof of (ii) is straightfoward. Finally, we verify (iii). Let A, $B \in \mathcal{CB}(X)$. Obviously, $\mathcal{D}_p(A,B) \ge 0$ and $\mathcal{D}_p(A,B)=\mathcal{D}_p(B,A)$. By (i), we have $\mathcal{D}_p(A,B)=0 \Longleftrightarrow A=B$. Applying (ii), we have

$$\begin{aligned}\mathcal{D}_p(A,B) &= \max\{\delta_p(A,B),\delta_p(B,A)\}\\ &\le \max\{\delta_p(A,C)+\delta_p(C,B),\ \delta_p(B,C)+\delta_p(C,A)\}\\ &\le \mathcal{D}_p(A,C)+\mathcal{D}_p(C,B).\end{aligned}$$

Therefore $\mathcal{D}_p(A,B)$ is a metric on $\mathcal{CB}(X)$. $\square$

2. Some new fixed point theorems

Let X and Y be two topological vector spaces. A multivalued map $L : X \to 2^Y$ is said to be *closed* if $G_r(L)=\{(x,y) \in X \times Y : x \in X, y \in L(x)\}$, the graph of L, is a closed subset of $X \times Y$. A point p in X is a fixed point of a map T if $p = Tp$ (if $T : X \to X$ is a single-valued map) or $p \in Tp$ (if $T : X \to 2^X$ is a multivalued map). The set of fixed points of a map T is denoted by $\mathcal{F}(T)$. Below, unless otherwise specified, let (X,d) be a complete metric space and $\mathcal{D}_p$ be a τ^0-metric on $\mathcal{CB}(X)$ induced by a τ^0-function p.

Theorem 2.1. Let $\alpha,\ \beta \geq 0$ with $\alpha+\beta < \frac{1}{2}$ and $T : X \to \mathcal{CB}(X)$ a multivalued map. Suppose that for each $x \in X$,

$$\mathcal{D}_p(Tx,Ty) \leq \alpha p(x,Tx) + \beta p(x,Ty) \text{ for all } y \in Tx$$

and T further satisfies one of the following conditions:

(D1) T is closed;

(D2) $\inf\{p(x,z)+p(x,Tx) : x \in X\} > 0$ for every $z \notin \mathcal{F}(T)$.

Then T has a fixed point in X.

Proof. Let $k = \frac{1}{2(1-\beta)}$ and take $u_0 \in X$. Then $k \in (0,1)$. If $u_0 \in Tu_0$, then we are done. If $u_0 \notin Tu_0$, then $p(u_0,Tu_0) > 0$ by Lemma 1.2. Choose $u_1 \in Tu_0$. If $u_1 \in Tu_1$, then u_1 is a fixed point of T. Otherwise, since

$$0 < p(u_1,Tu_1) \leq \mathcal{D}_p(Tu_0,Tu_1) \leq \alpha p(u_0,Tu_0) + \beta p(u_0,Tu_1),$$

there exists $u_2 \in Tu_1$ such that

$$\begin{aligned} p(u_1,u_2) &< (\frac{1}{2}-\beta)p(u_0,u_1) + \beta p(u_0,u_2) \\ &\leq \frac{1}{2}p(u_0,u_1) + \beta p(u_1,u_2) \end{aligned}$$

and hence

$$0 < p(u_1,u_2) < kp(u_0,u_1).$$

If $u_2 \in Tu_2$, then u_2 is a fixed point of T. Otherwise, there exists $u_3 \in Tu_2$ such that $0 < p(u_2,u_3) < kp(u_1,u_2)$. Continue in this way, we can construct inductively a sequence $\{u_n\}_{n=0}^{\infty}$ in X satisfying $u_n \in T(u_{n-1})$, $p(u_{n-1},u_n) > 0$ and $p(u_n,u_{n+1}) < kp(u_{n-1},u_n)$ for each $n \in \mathbb{N}$. It follows that

$$p(u_n,u_{n+1}) < k^n p(u_0,u_1)$$

for each $n \in \mathbb{N}$. Let $\beta_n = \frac{k^n}{1-k}p(u_0,u_1)$. For $m,n \in \mathbb{N}$ with $m > n$, we have

$$p(u_n,u_m) \leq \sum_{j=n}^{m-1} p(u_j,u_{j+1}) \leq \beta_n. \qquad (**)$$

Since $0 < k < 1$, $\lim_{n\to\infty} \sup\{p(u_n, u_m) : m > n\} = 0$. By Lemma 1.1 and the completeness of X, $\{u_n\}$ is a Cauchy sequence and there exists $v_0 \in X$ such that $u_n \to v_0$. From (τ2) and ($**$), we have

$$p(u_n, v_0) \leq \beta_n \text{ for all } n \in \mathbb{N}.$$

We claim that $v_0 \in Tv_0$. If (D1) holds, i.e. T is closed, since $u_n \in T(u_{n-1})$ and $u_n \to v_0$, we have $v_0 \in Tv_0$. Finally, let (D2) holds. Suppose not, we have

$$\begin{aligned} 0 &< \inf_{x\in X}\{p(x, v_0) + p(x, Tx)\} \\ &\leq \inf_{n\in\mathbb{N}}\{p(u_n, v_0) + p(u_n, u_{n+1})\} \leq \lim_{n\to\infty} 2\beta_n = 0, \end{aligned}$$

which is a contradiction. Therefore $v_0 \in Tv_0$ and we complete the proof. □

The following conclusions are immediate from Theorem 4.1.

Corollary 2.1. Let α, $\beta \geq 0$ with $\alpha + \beta < \frac{1}{2}$ and $T : X \to \mathcal{CB}(X)$ a multivalued map. Suppose that for each $x \in X$,

$$\mathcal{H}(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(x, Ty) \text{ for all } y \in Tx$$

and T further satisfies one of the following conditions:

(H1) T is closed.;

(H2) $\inf\{d(x, z) + d(x, Tx) : x \in X\} > 0$ for every $z \notin \mathcal{F}(T)$.

Then T has a fixed point in X.

Remark 2.1. In Corollary 2.1, if the condition (H2) is replaced with the condition (H3), then the conclusion is still true, where

(H3) $\inf\{d(x, z) + d(z, Tx) : x \in X\} > 0$ for every $z \notin \mathcal{F}(T)$.

Corollary 2.2. Let α, $\beta \geq 0$ with $\alpha + \beta < \frac{1}{2}$. Suppose that $T : X \to X$ is a map satisfying

(i) $d(Tx, T^2x) \leq \alpha d(x, Tx) + \beta d(x, T^2x)$ for all $x \in X$;

(ii) $\inf\{d(x, z) + d(x, Tx) : x \in X\} > 0$ for every $z \in X$ with $z \neq Tz$.

Then T has a fixed point in X.

Corollary 2.3. Let $(X, \|\cdot\|)$ be a Banach space and $\alpha, \beta \geq 0$ with $\alpha+\beta < \frac{1}{2}$. Suppose that $T : X \to X$ is a continuous map satisfying $\|Tx - T^2x\| \leq \alpha \|x - Tx\| + \beta \|x - T^2x\|$ for all $x \in X$. Then T has a fixed point in X.

Proof. Let $\varphi : X \to 2^X$ be defined by $\varphi(x) = \{T(x)\}$ for all $x \in X$. Then $\varphi(x) \in \mathcal{CB}(X)$ for all $x \in X$ and φ is u.s.c.. Note that for each $x \in X$,

$$\mathcal{H}(\varphi x, \varphi y) = \left\|Tx - T^2x\right\| \leq \alpha \|x - \varphi x\| + \beta \|x - \varphi y\|$$

for all $y \in \varphi(x) = \{T(x)\}$. Since φ is u.s.c. with closed values, φ is closed. Hence the conclusion follows from Theorem 2.1. □

Theorem 2.2. Let $T : X \to \mathcal{CB}(X)$ be a multivalued map and $0 \leq \alpha < 1$. Suppose that for each $x \in X$,

$$\mathcal{D}_p(Tx, Ty) \leq \alpha p(x, Tx) \text{ for all } y \in Tx$$

and T further satisfies one of conditions (D1) and (D2) as in Theorem 2.1. Then T has a fixed point in X.

Proof. Let $\gamma \in (\alpha, 1)$ and take $u_0 \in X$. If $u_0 \in Tu_0$, then we are done. If $u_0 \notin Tu_0$, then $p(u_0, Tu_0) > 0$ by Lemma 1.2. Choose $u_1 \in Tu_0$. If $u_1 \in Tu_1$, then u_1 is a fixed point of T. Otherwise, since

$$0 < p(u_1, Tu_1) \leq \mathcal{D}_p(Tu_0, Tu_1) < \gamma p(u_0, Tu_0)$$

there exists $u_2 \in Tu_1$ such that $p(u_1, u_2) < \gamma p(u_0, u_1)$. Hence we can construct a sequence $\{u_n\}_{n=0}^{\infty}$ in X satisfying $u_n \in T(u_{n-1})$, $p(u_{n-1}, u_n) > 0$ and $p(u_n, u_{n+1}) < \gamma p(u_{n-1}, u_n)$ for each $n \in \mathbb{N}$. By the similar argument as in the proof of Theorem 2.1, we can prove the theorem. □

Corollary 2.4. Let $(X, \|\cdot\|)$ be a Banach space and $0 \leq \alpha < 1$. Suppose that $T : X \to X$ is a continuous map satisfying $\|Tx - T^2x\| \leq \alpha \|x - Tx\|$ for all $x \in X$. Then T has a fixed point in X.

3. Generalized Kannan's and Chatterjea's theorems

Recall that a selfmap $T : X \to X$ is said to be

(a) Kannan's type [2,3,7] if there exists $\gamma \in [0, \frac{1}{2})$, such that $d(Tx, Ty) \leq \gamma\{d(x, Tx) + d(y, Ty)\}$ for all x, $y \in X$;

(b) Chatterjea's type [1] if there exists $\gamma \in [0, \frac{1}{2})$, such that $d(Tx, Ty) \leq \gamma\{d(x, Ty) + d(y, Tx)\}$ for all x, $y \in X$.

In this section, we will first establish some new fixed point theorems by using Theorem 2.2.

Theorem 3.1. Let $T : X \to \mathcal{CB}(X)$ be a multivalued map and $0 \leq \gamma < \frac{1}{2}$. Suppose that

$$\mathcal{D}_p(Tx, Ty) \leq \gamma\{p(x, Tx) + p(y, Tx) + p(y, Ty)\} \text{ for all } x, y \in X$$

and T further satisfies one of conditions (D1) and (D2) as in Theorem 2.1. Then T has a fixed point in X.

Proof. Let $x \in X$ and put $k = \frac{\gamma}{1-\gamma}$. Then $k \in [0, 1)$. Let $y \in Tx$ be arbitrary. Then $p(y, Tx) = 0$. Since

$$\begin{aligned} \mathcal{D}_p(Tx, Ty) &\leq \gamma\{p(x, Tx) + p(y, Tx) + p(y, Ty)\} \\ &\leq \gamma\{p(x, Tx) + \mathcal{D}_p(Tx, Ty)\}, \end{aligned}$$

we have $\mathcal{D}_p(Tx, Ty) \leq kp(x, Tx)$ for all $y \in Tx$. Therefore the conclusion follows from Theorem 2.2. □

Corollary 3.1. Let $0 \leq \gamma < \frac{1}{2}$. Suppose that $T : X \to X$ is a map satisfying $d(Tx, Ty) \leq \gamma\{d(x, Tx) + d(y, Tx) + d(y, Ty)\}$ for all x, $y \in X$, then T has a unique fixed point in X.

Proof. Clearly, the uniqueness of fixed point is true. It suffices to show that $\inf\{d(x, z) + d(x, Tx) : x \in X\} > 0$ for every $z \notin \mathcal{F}(T)$. Assume that there exists $w \in X$ with $w \neq Tw$ and $\inf\{d(x, w) + d(x, Tx) : x \in X\} = 0$. Then there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty}\{d(u_n, w)+d(u_n, Tu_n)\} = 0$. It follows that $d(u_n, w) \to 0$ and $d(u_n, Tu_n) \to 0$ and hence $d(w, Tu_n) \to 0$ as $n \to \infty$. By hypothesis, we have

$$d(Tu_n, Tw) \leq \gamma\{d(u_n, Tu_n) + d(w, Tu_n) + d(w, Tw)\}$$

for all $n \in \mathbb{N}$. Let $n \to \infty$, we have $d(w, Tw) \leq \gamma d(w, Tw)$, which is a contradiction. Therefore the conclusion follows from Theorem 3.1. □

Some fixed point theorems for multivalued maps of generalized Kannan's type can be established immediately from above theorems.

Theorem 3.2. (Generalized Kannan's fixed point theorem for multivalued maps) Let $T : X \to \mathcal{CB}(X)$ be a multivalued map and $0 \leq \gamma < \frac{1}{2}$. Suppose that

$$\mathcal{D}_p(Tx, Ty) \leq \gamma\{p(x, Tx) + p(y, Ty)\} \text{ for all } x, y \in X$$

and T further satisfies one of conditions (D1) and (D2) as in Theorem 2.1. Then T has a fixed point in X.

In [2, Corollary 3], it have shown that if $T : X \to X$ is a map of Kannan's type, then $\inf\{d(x,z) + d(x,Tx) : x \in X\} > 0$ for every $z \in X$ with $z \neq Tz$.

Corollary 3.2. **[2,3,7]** Suppose that $T : X \to X$ is a map of Kannan's type, then T has a unique fixed point in X.

Applying Theorem 2.1, we can establish some fixed point theorems for multivalued maps of generalized Chatterjea's type.

Theorem 3.3. **(Generalized Chatterjea's fixed point theorem for multivalued mappings)** Let $T : X \to \mathcal{CB}(X)$ be a multivalued map and $0 \leq \gamma < \frac{1}{2}$. Suppose that

$$\mathcal{D}_p(Tx, Ty) \leq \gamma\{p(x, Ty) + p(y, Tx)\} \text{ for all } x, y \in X$$

and T further satisfies one of conditions (D1) and (D2) as in Theorem 2.1. Then T has a fixed point in X.

Proof. For each $x \in X$, let $y \in Tx$ be arbitrary. Since $p(y, Tx) = 0$, we have $\mathcal{D}_p(Tx, Ty) \leq \gamma p(x, Ty)$ for all $y \in Tx$. Therefore the conclusion follows from Theorem 2.1. □

Corollary 3.3. **[1]** Suppose that $T : X \to X$ is a map of Chatterjea's type, then T has a unique fixed point in X.

Proof. Obviously, the uniqueness of fixed point is true. Using the similar argument as in Corollary 3.1, we can show that $\inf\{d(x,z) + d(x,Tx) : x \in X\} > 0$ for every $z \in X$ with $z \neq Tz$ and hence the conclusion follows from Theorem 3.3. □

Using the concept of τ-function, we establish the following fixed point theorem. In the following theorem, note that we don't assume that $p(x,x) = 0$ for all $x \in X$.

Theorem 3.4. Let p be a τ-function on X. Let $0 \leq \gamma < \frac{1}{2}$. Suppose that $T : X \to X$ is a map satisfying

(i) $p(Tx, T^2x) \leq \gamma p(x, T^2x)$ for all $x \in X$;

(ii) $\inf\{p(x,z) + p(x,Tx) : x \in X\} > 0$ for every $z \in X$ with $z \neq Tz$.

Then T has a fixed point in X. Moreover, if v is a fixed point of T, then $p(v,v) = 0$.

Proof. Let $u \in X$. Define $u_0 = u$ and $u_n = T^n u_0$ for every $n \in \mathbb{N}$. Put $k = \frac{\gamma}{1-\gamma}$. Then $0 \leq k < 1$. By hypothesis, we have

$$p(u_n, u_{n+1}) \leq \gamma p(u_{n-1}, u_{n+1}) \leq \gamma\{\, p(u_{n-1}, u_n) \; + p(u_n, u_{n+1})\, \},$$

for all $n \in \mathbb{N}$. It follows that

$$p(u_n, u_{n+1}) \leq kp(u_{n-1}, u_n) \leq \cdots \leq k^n p(u_0, u_1)$$

for all $n \in \mathbb{N}$. Using the similar argument as in the proof of Theorem 2.1, we can prove that there exist a Cauchy sequence $\{u_n\}$ and v_0 in X such that $u_n \to v_0$ as $n \to \infty$. Also, we have $v_0 \in \mathcal{F}(T)$. Since

$$p(v_0, v_0) = p(Tv_0, T^2v_0) \leq \gamma p(v_0, T^2v_0) = \gamma p(v_0, v_0),$$

we have $p(v_0, v_0) = 0$. The proof is completed. □

Remark 3.1.

(a) In Theorem 3.4, the τ-function p do not need to satisfy $p(x,x) = 0$ for all $x \in X$. Hence Theorem 3.4 is not a special case of Theorem 2.1;

(b) Condition (i) of Theorem 3.4 is different from the condition of T in [2, Theorem 4]. In [2, Theorem 4], T is assumed to satisfy $p(Tx, T^2x) \leq \gamma p(x, Tx)$, $0 \leq \gamma < \frac{1}{2}$, for all $x \in X$.

Lemma 3.1. [2,5] Let (X, d) be a metric space and p a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0. If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.

Applying Theorem 3.4, we obtain the following fixed point Theorem. In the following result, p is a w-distance on X.

Theorem 3.5. Let p be a w-distance on X and $0 \leq \gamma < \frac{1}{2}$. Suppose that $T : X \to X$ is a continuous map satisfying $p(Tx, T^2x) \leq \gamma p(x, T^2x)$ for all $x \in X$. Then T has a fixed point in X. Moreover, if v is a fixed point of T, then $p(v,v) = 0$.

Proof. It suffices to show that $\inf\{p(x,z)+p(x,Tx) : x \in X\} > 0$ for every $z \in X$ with $z \neq Tz$. Assume that there exists $w \in X$ with $w \neq Tw$ and $\inf\{p(x,w)+p(x,Tx) : x \in X\} = 0$. Then there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty}\{p(u_n,w)+p(u_n,Tu_n)\} = 0$. It follows that $p(u_n,w) \to 0$ and $p(u_n,Tu_n) \to 0$ as $n \to \infty$. By Lemma 3.1, we have $Tu_n \to w$. On the other hand, by assumption, we have

$$\begin{aligned} p(u_n, T^2u_n) &\leq p(u_n, Tu_n) + p(Tu_n, T^2u_n) \\ &\leq p(u_n, Tu_n) + \gamma p(u_n, T^2u_n) \end{aligned}$$

and hence

$$p(u_n, T^2u_n) \leq \frac{1}{1-\gamma} p(u_n, Tu_n)$$

for all $n \in \mathbb{N}$. Thus $p(u_n, T^2u_n) \to 0$ as $n \to \infty$. By Lemma 3.1 again, we have $T^2u_n \to w$. Since $T : X \to X$ is continuous, we have

$$T(w) = T(\lim_{n\to\infty} Tu_n) = \lim_{n\to\infty} T^2u_n = w,$$

which is a contradiction. Therefore, using Theorem 3.4, there exists $v_0 \in X$ such that $v_0 = Tv_0$ and $p(v_0, v_0) = 0$. This completes the proof. □

Remark 3.2. In [2, Corollary 2], T is assumed to satisfy $p(Tx, T^2x) \leq \gamma p(x, Tx)$, $0 \leq \gamma < \frac{1}{2}$, for all $x \in X$. Hence Theorem 3.5 is different from [2, Corollary 2].

Finally, we can obtain the following interesting fixed point theorems by applying Theorem 2.1.

Theorem 3.6. Let $T : X \to \mathcal{CB}(X)$ be a multivalued map and $0 \leq \gamma < \frac{1}{5}$. Suppose that

$$\mathcal{D}_p(Tx, Ty) \leq \gamma\{p(x, Tx) + p(x, Ty) + p(y, Tx) + p(y, Ty)\} \text{ for all } x, y \in X$$

and T further satisfies one of conditions (D1) and (D2) as in Theorem 2.1. Then T has a fixed point in X.

Proof. Let $x \in X$ and set $\alpha = \beta = \frac{\gamma}{1-\gamma} \geq 0$. Then $\alpha + \beta < \frac{1}{2}$. Let $y \in Tx$ be arbitrary. Then $p(y, Tx) = 0$. Since

$$\begin{aligned} \mathcal{D}_p(Tx, Ty) &\leq \gamma\{p(x, Tx) + p(x, Ty) + p(y, Tx) + p(y, Ty)\} \\ &\leq \gamma\{p(x, Tx) + p(x, Ty) + \mathcal{D}_p(Tx, Ty)\}, \end{aligned}$$

we have $\mathcal{D}_p(Tx, Ty) \leq \alpha p(x, Tx) + \beta p(x, Ty)$ for all $y \in Tx$. Therefore the conclusion follows from Theorem 2.1. □

Theorem 3.7. Let α, $\beta \geq 0$ with $2\alpha + 3\beta < 1$ and $T : X \to \mathcal{CB}(X)$ a multivalued map satisfying one of conditions (D1) and (D2) as in Theorem 2.1. Suppose that one of the following conditions hold:

(L1) for each $x \in X$, $\mathcal{D}_p(Tx, Ty) \leq \alpha\{p(x, Tx) + p(y, Tx)\} + \beta\{p(x, Ty) + p(y, Ty)\}$ for all $y \in Tx$;

(L2) for each $x \in X$, $\mathcal{D}_p(Tx, Ty) \leq \alpha\{p(x, Ty) + p(y, Tx)\} + \beta\{p(x, Tx) + p(y, Ty)\}$ for all $y \in Tx$;

(L3) for each $x \in X$, $\mathcal{D}_p(Tx, Ty) \leq \alpha p(x, Tx) + \beta\{p(x, Ty) + p(y, Tx) + p(y, Ty)\}$ for all $y \in Tx$;

(L4) for each $x \in X$, $\mathcal{D}_p(Tx, Ty) \leq \alpha p(x, Ty) + \beta\{p(x, Tx) + p(y, Tx) + p(y, Ty)\}$ for all $y \in Tx$.

Then T has a fixed point in X.

Proof. We only prove that the conclusion holds when (L1) is satisfied. The rest of the proof is similar to this. Let $x \in X$. Take $\alpha_1 = \frac{\alpha}{1-\beta} \geq 0$ and $\beta_1 = \frac{\beta}{1-\beta} \geq 0$. Then $\alpha + \beta < \frac{1}{2}$. Let $y \in Tx$ be arbitrary. Then $p(y, Tx) = 0$ and hence $\mathcal{D}_p(Tx, Ty) \leq \alpha p(x, Tx) + \beta p(x, Ty)$ for all $y \in Tx$. Therefore the conclusion also follows from Theorem 2.1. □

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