

Skew-Hadamard Matrices of Orders 188 and 388 Exist

Dragomir Ž. Đoković¹

Department of Pure Mathematics, University of Waterloo
Waterloo, Ontario, N2L 3G1, Canada
djokovic@uwaterloo.ca

Abstract. We construct several difference families on cyclic groups of orders 47 and 97, and use them to construct skew-Hadamard matrices of orders 188 and 388. Such difference families and matrices are constructed here for the first time. The matrices are constructed by using the Goethals–Seidel array.

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1. INTRODUCTION

Recall that a Hadamard matrix A of order m is a $\{\pm 1\}$ -matrix of size $m \times m$ such that $AA^T = mI_m$, where T denotes the transpose and I_m the identity matrix. A skew-Hadamard matrix is a Hadamard matrix A such that $A - I_m$ is a skew-symmetric matrix. We refer the reader to [1] for the survey of known results about skew-Hadamard matrices.

The construction of skew-Hadamard matrices is lagging considerably behind that for arbitrary Hadamard matrices. Our previous four notes, written more than 13 years ago, were motivated by the desire to improve this situation. We constructed skew-Hadamard matrices of order $m = 4n$ for the following 24 odd integers n :

- [2]: 37, 43;
- [3]: 67, 113, 127, 157, 163, 181, 241;
- [4]: 39, 49, 65, 93, 121, 129, 133, 217, 219, 267;
- [6]: 81, 103, 151, 169, 463.

At the time of publication, such matrices of these orders were not known to exist. Due to the manifold increase in computing power since that time, one can now make further progress.

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In [6], we listed 45 odd integers $n < 300$ for which no skew-Hadamard matrix of order $4n$ was known at that time. (In the first edition of [1], Table 24.31 was incomplete.) The smallest of these n 's was 47. The next one, 59, has been removed recently by Fletcher, Koukouvinos and Seberry [7]. In this note we shall remove the integers 47 and 97 from the mentioned list by constructing examples of skew-Hadamard matrices of orders $4 \cdot 47 = 188$ and $4 \cdot 97 = 388$. (We have constructed a bunch of examples but we have saved and will present only a few of them.)

Consequently, the revised list now consists of the 42 integers:

69, 89, 101, 107, 109, 119, 145, 149, 153, 167, 177, 179, 191, 193,
 201, 205, 209, 213, 223, 225, 229, 233, 235, 239, 245, 247, 249, 251,
 253, 257, 259, 261, 265, 269, 275, 277, 283, 285, 287, 289, 295, 299.

We construct our examples of skew-Hadamard matrices of orders 188 and 388 by constructing first suitable supplementary difference sets, and then we use these sets to build four circulant blocks, which one should plug into the Goethals–Seidel array. The procedure used to find these supplementary difference sets is not new. I have used it in several papers during the last 15 years. It is described in my note [5].

2. THE CASE $n = 47$

We denote the additive group of integers modulo n by \mathbf{Z}_n . In this section we set $n = 47$. In the literature on Hadamard matrices it is customary to refer to difference families (DF) as supplementary difference sets (SDS) and to employ more elaborate and more informative notation by listing the order v of the underlying abelian group, the number of sets in the family as well as their cardinals, and also the parameter λ .

We have constructed four suitable difference families in \mathbf{Z}_n . The first two are the following.

Proposition 2.1. *Define six subsets of \mathbf{Z}_{47} :*

$$X_1 = \{2, 3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 25, 27, 30, \\ 31, 33, 35, 37, 38, 39, 40, 42, 43, 44\},$$

$$X_2 = \{1, 3, 6, 7, 8, 11, 13, 14, 15, 19, 20, 21, 24, 27, 30, 33, 39, 41, 43, \\ 44, 45, 46\},$$

$$X_3 = \{3, 6, 8, 10, 11, 12, 14, 20, 21, 23, 24, 25, 26, 27, 30, 31, 32, 34, 35, \\ 41, 42, 45\},$$

$$Y_1 = \{1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14, 15, 17, 18, 19, 21, 23, 24, 25, 27, \\ 28, 29, 30, 31, 35, 38, 41, 43, 44, 46\},$$

$$Y_2 = \{3, 6, 7, 8, 10, 11, 12, 16, 22, 25, 26, 31, 32, 33, 34, 37, 39, 41, 42, \\ 43, 44, 46\},$$

$$Y_3 = \{3, 7, 12, 13, 15, 16, 18, 20, 21, 23, 25, 26, 27, 28, 32, 35, 38, 39, 42, \\ 44, 45, 46\}.$$

The triples $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$ are difference families, i.e., they are $3 - (47; 30, 22, 22; 39)$ supplementary difference sets in \mathbf{Z}_{47} . The two families are not equivalent.

Proof. Use the computer to verify the claims. Note that the cardinals $n_k = |X_k| = |Y_k|$ are indeed $n_1 = 30$ and $n_2 = n_3 = 22$. The parameter λ is 39, i.e., each nonzero integer in \mathbf{Z}_n occurs 39 times in the list of differences created from the sets X_k and also from the Y_k .

The second claim can be verified in several ways. We used the following ad hoc method. We compare the list of differences generated by the sets X_1 and Y_1 . Each nonzero integer $i \in \mathbf{Z}_n$ occurs in one of these lists say μ_i times. The μ_i 's take only three values: 18, 19 or 20. But the number of μ_i 's equal to 18, 19 and 20 is 12, 26 and 8 for X_1 and 14, 22 and 10 for Y_1 . Hence X_1 and Y_1 are not equivalent under translations and automorphisms of the additive group \mathbf{Z}_n . \square

For any subset $X \subseteq \mathbf{Z}_n$ let

$$a_X = (a_0, a_1, \dots, a_{n-1})$$

be the $\{\pm 1\}$ -row vector such that $a_i = -1$ iff $i \in X$. We denote by A_X the $n \times n$ circulant matrix having a_X as its first row.

Let $X_0 \subseteq \mathbf{Z}_n$ be the Paley difference set (the set of nonzero squares in the finite field \mathbf{Z}_n). Recall that X_0 is of skew type, i.e., for nonzero $i \in \mathbf{Z}_n$ we have $i \in X_0$ iff $-i \notin X_0$. Its cardinal is $n_0 = |X_0| = 23$.

For simplicity, write A_k instead of A_{X_k} for $k = 0, 1, 2, 3$. We can now plug our matrices A_k into the Goethals–Seidel template to construct a skew-Hadamard

matrix of order 188:

$$A = \begin{bmatrix} A_0 & A_1R & A_2R & A_3R \\ -A_1R & A_0 & -A_3^T R & A_2^T R \\ -A_2R & A_3^T R & A_0 & -A_1^T R \\ -A_3R & -A_2^T R & A_1^T R & A_0 \end{bmatrix}.$$

As usual, R denotes the matrix having ones on the back-diagonal and all other entries zero.

Clearly, we can use the second difference family to construct another skew-Hadamard matrix of order 188. Both solutions have the same associated decomposition of $4n$ as sum of four squares:

$$\begin{aligned} 4n = 188 &= 13^2 + 3^2 + 3^2 + 1^2 \\ &= \sum_{k=0}^3 (n - 2n_k)^2. \end{aligned}$$

The remaining two difference families have different parameters from the first two.

Proposition 2.2. *Define six subsets of \mathbf{Z}_{47} :*

$$P_1 = \{0, 2, 4, 5, 9, 10, 12, 16, 17, 19, 21, 22, 23, 25, 27, 28, 35, 36, 37, 43, 46\},$$

$$P_2 = \{0, 1, 2, 6, 8, 9, 11, 15, 16, 19, 25, 32, 33, 35, 36, 37, 38, 40, 44\},$$

$$P_3 = \{1, 2, 3, 4, 5, 6, 7, 10, 11, 16, 18, 22, 24, 28, 31, 35, 38, 40, 43\},$$

$$Q_1 = \{4, 5, 6, 8, 11, 12, 15, 20, 21, 23, 25, 26, 28, 29, 30, 31, 32, 36, 39, 41, 43\},$$

$$Q_2 = \{1, 2, 5, 7, 13, 14, 21, 22, 24, 26, 31, 32, 35, 36, 37, 39, 40, 42, 46\},$$

$$Q_3 = \{1, 2, 3, 4, 5, 9, 12, 18, 20, 21, 24, 25, 32, 34, 38, 39, 43, 44, 46\}.$$

The triples $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ are difference families, i.e., they are $3 - (47; 21, 19, 19; 24)$ supplementary difference sets in \mathbf{Z}_{47} . These two families are not equivalent to each other or the ones above.

Just as the first two families, $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ can be used to construct two more skew-Hadamard matrices of order 188. The associated decomposition into sum of four squares is now different: $188 = 9^2 + 9^2 + 5^2 + 1^2$.

3. THE CASE $n = 97$

For the remainder of this note we set $n = 97$. Let G be the multiplicative group of the nonzero elements of \mathbf{Z}_n , a cyclic group of order $n - 1 = 96$, and let $H = \langle 35 \rangle = \{1, 35, 61\}$ be its subgroup of order 3. We use the same enumeration of the 32 cosets α_i , $0 \leq i \leq 31$, of H in G as in our computer

program. Thus we impose the condition that $\alpha_{2i+1} = -1 \cdot \alpha_{2i}$ for $0 \leq i \leq 15$. For even indices we have

$$\begin{aligned} \alpha_0 &= H, & \alpha_2 &= 2H, & \alpha_4 &= 3H, & \alpha_6 &= 4H, & \alpha_8 &= 5H, \\ \alpha_{10} &= 6H, & \alpha_{12} &= 7H, & \alpha_{14} &= 9H, & \alpha_{16} &= 10H, & \alpha_{18} &= 12H, \\ \alpha_{20} &= 13H, & \alpha_{22} &= 15H, & \alpha_{24} &= 18H, & \alpha_{26} &= 20H, & \alpha_{28} &= 23H, \\ \alpha_{30} &= 26H. \end{aligned}$$

Next define four index sets:

$$\begin{aligned} J_0 &= \{1, 2, 4, 6, 9, 11, 13, 14, 17, 18, 21, 23, 25, 27, 29, 30\}, \\ J_1 &= \{1, 2, 6, 7, 8, 9, 10, 11, 12, 13, 23, 27, 29\}, \\ J_2 &= \{0, 1, 2, 5, 6, 12, 13, 15, 16, 20, 24, 25, 26, 29, 30, 31\}, \\ J_3 &= \{0, 2, 3, 4, 7, 8, 9, 11, 12, 13, 15, 16, 17, 18, 23, 28, 29\} \end{aligned}$$

and introduce the following four subsets of \mathbf{Z}_n :

$$U_k = \bigcup_{i \in J_k} \alpha_i, \quad k = 0, 1, 2, 3.$$

Their cardinals $n_k = |U_k| = 3|J_k|$ are:

$$n_0 = n_2 = 48, \quad n_1 = 39, \quad n_3 = 51$$

and we set

$$\lambda = n_0 + n_1 + n_2 + n_3 - n = 89.$$

Observe that U_0 is of skew type, i.e., we have

$$U_0 \cap (-U_0) = \emptyset, \quad U_0 \cup (-U_0) = \mathbf{Z}_n \setminus \{0\}.$$

Proposition 3.1. *The four subsets $U_0, U_1, U_2, U_3 \subset \mathbf{Z}_n$ form a difference family, i.e., they are $4-(97; 48, 39, 48, 51; 89)$ supplementary difference sets in \mathbf{Z}_{97} .*

Proof. For $r \in \{1, 2, \dots, 96\}$ let $\lambda_k(r)$ denote the number of solutions of the congruence $i - j \equiv r \pmod{97}$ with $\{i, j\} \subseteq U_k$. It is easy to verify (by using a computer) that

$$\lambda_1(r) + \lambda_2(r) + \lambda_3(r) + \lambda_4(r) = \lambda$$

is valid for all such r . Hence the sets U_1, U_2, U_3, U_4 form a difference family in \mathbf{Z}_n . □

Let A_k now denote the $n \times n$ circulant matrices A_{Y_k} . The SDS-property implies that the $\{\pm 1\}$ -matrices A_0, \dots, A_3 satisfy the identity

$$\sum_{k=0}^3 A_k A_k^T = 4nI_n.$$

One can now plug the matrices A_k into the Goethals–Seidel template to obtain a Hadamard matrix A of order $4n = 388$. Since U_1 is of skew type, A is also skew-Hadamard.

Our second example, B , is constructed in the same way by using the index sets:

$$\begin{aligned} K_0 &= \{0, 3, 4, 7, 9, 11, 12, 14, 17, 19, 20, 22, 24, 27, 28, 30\}, \\ K_1 &= \{4, 7, 8, 10, 12, 13, 14, 15, 17, 18, 20, 26, 27\}, \\ K_2 &= \{0, 1, 2, 3, 6, 7, 8, 11, 12, 14, 20, 23, 24, 25, 28, 31\}, \\ K_3 &= \{1, 2, 4, 7, 8, 9, 10, 12, 13, 19, 21, 23, 24, 25, 26, 27, 31\}, \end{aligned}$$

with the corresponding subsets of \mathbf{Z}_n :

$$V_k = \bigcup_{i \in K_k} \alpha_i, \quad k = 0, 1, 2, 3,$$

with V_0 of skew type.

Proposition 3.2. *The four subsets $V_0, V_1, V_2, V_3 \subset \mathbf{Z}_n$ form a difference family, i.e., they are $4 - (97; 48, 39, 48, 51; 89)$ supplementary difference sets in \mathbf{Z}_{97} .*

The two SDS's that we used to construct A and B are not equivalent. For instance, the sets U_1 and V_1 are not equivalent under translations and group automorphisms of \mathbf{Z}_n .

Since the two SDS's have the same parameters, they share the same decomposition of $4n$ into sum of four squares:

$$\begin{aligned} 4n = 388 &= 19^2 + 5^2 + 1^2 + 1^2 \\ &= \sum_{k=1}^4 (n - 2n_k)^2. \end{aligned}$$

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