International Mathematical Forum, 3, 2008, no. 23, 1135 - 1140

Infinite-Dimensional "Compact Complex Manifolds"

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Abstract. Here we define and study infinite products of compact complex spaces as local ringed spaces and extend to them a few properties of the finite-dimensional case. The compactness of the factors is essential.

Mathematics Subject Classification: 32K99; 30F99; 32J99; 32L10

Keywords: infinite-dimensional complex manifold; compact complex manifold; Riemann surface

1. Introduction

Let I be a non-empty set. For every $i \in I$ let X_i be a reduced and connected compact complex space (finite-dimensional!). Set $X := X_I := \prod_{i\in I} X_i$. We use on X the product topology. Thus X is a compact and Hausdorff topological space. Notice that X is locally compact. We want to see X as a complex space and hence we want to define a sheaf of local $\mathbb{C}$-algebras $\mathcal{O}_X$ on X. For any $S \subseteq I$ set $X_S := \prod_{i\in S} X_i$ and let $\pi_S : X \to X_S$ be the projection. There is a basis $\mathfrak{B}$ of open subsetes of X such that for every $U \in \mathfrak{B}$ there is a finite $S \subseteq X$ and an open subset V of X_S such that $U = \pi_S^{-1}(V)$. Set $\mathcal{O}_X(U) := \mathcal{O}_{X_S}(V)$. The $\mathbb{C}$-algebra $\mathcal{O}_X(U)$ is well-defined (i.e. it does not depend from the choice of S), because for any $i \in I$ every holomorphic function on X_i is constant. Similarly, for any open subset W of U such that $W \in \mathfrak{B}$ we may define a restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(W)$. Let $\mathcal{O}_X$ be the sheaf associated to the $\mathbb{C}$-algebras $\mathcal{O}_X(U)$, $U \in \mathfrak{B}$. We are mainly interested in the case " X_i is smoooth for all i ". In this case we want to see X as a complex manifold. If $\dim(X_i) \geq 1$ for infinitely many $i \in I$, then X is not a finite-dimensional topological space for any reasonable topological notion of dimension ([2], Ch. 7). If each X_i is smooth and $\dim(X_i) \geq 1$ for infinitely many $i \in I$, then

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[1] The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

X cannot be locally modelled on open subsets of a topological vector space, because every Hausdorff topological vector space with a compact neighborhood of 0 is finite-dimensional. In section 2 we will list a few elementary properties of these objects. A morphism h between to such products $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is just a morphism in the category of local $\mathbb{C}$-ringed spaces. For any complex space $(A, \mathcal{O}_A)$ (not necessarily compact) and any $(X, \mathcal{O}_X)$ as above, a morphism $h : (A, \mathcal{O}_A) \to (X, \mathcal{O}_X)$ is a morphism in the category of local $\mathbb{C}$-ringed spaces. As a test for our set-up in section 2 we will prove the following result related to the following problem, called the simplification problem: if $Z \times X \cong Z \times Y$, is it true that $X \cong Y$?

Theorem 1. *Let Z be either $\mathbf{P}^n$, $n \geq 1$, or a smooth and connected compact Riemann surface of genus $g \geq 2$. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be arbitrary products of smooth and connected compact complex manifolds. If $Z \times X \cong Z \times Y$ (as local ringed spaces), then $X \cong Y$.*

In section 3 we will define " an almost metric " on X mimicking the definition of the Kobayashi pseudometric ([3]). In section 4 we will consider products of smooth, compact and connected Riemann surfaces.

2. MAIN DEFINITIONS AND PROOF OF THEOREM 1

Set $X := X_I := \prod_{i \in I} X_i$ with each X_i a reduced, connected and compact complex space.

Fix an integer $r \geq 1$. A rank r vector bundle E on X is a rank r locally free $\mathcal{O}_X$-sheaf on X. If $r = 1$ we will say that E is a line bundle.

Lemma 1. *Let E be a holomorphic vector bundle on X. Then there is a finite subset S of I and a holomorphic vector bundle F on X_S such that $E \cong \pi_S^*(F)$.*

Proof. Since X is compact, E has a finite trivializing atlas, say $\{U_j\}_{1 \leq j \leq n}$, formed by basic open sets, i.e. for each $j \in \{1, \ldots, n\}$ there is a finite subset S_j of I and an open $V_j \subseteq X_{S_j}$ such that $U_j = \pi_{S_j}^{-1}(V_j)$. Obviously, we may take $S = \cup_{j=1}^n S_j$. □

Proposition 1. *Let E be a holomorphic vector bundle on X. Then $\dim(H^0(X, E)) < +\infty$.*

Proof. Take S and F as in Lemma 1. Since each X_i is compact, reduced and connected, it is easy to check that $H^0(X, E) \cong H^0(X_S, F)$ and hence $\dim(H^0(X, E)) < +\infty$. □

Remark 1. It is easy to check that Proposition 1 does not extend to higher cohomology groups. For instance, it is easy to see that $\dim(H^1(X, \mathcal{O}_X)) = +\infty$ if $H^1(X_i, \mathcal{O}_{X_i}) \neq 0$ for infinitely many $i \in I$. Furthermore, it is easy to give examples of products with $H^k(X, \mathcal{O}_X) \neq 0$ for all $k \in \mathbb{N}$.

Now we define in the usual way the notion of Cartier divisor. Fix any open covering $\{U_j\}_{j \in J}$ of X. For any $j \in J$ let f_j, h_j be elements of $\mathcal{O}_X(U_j)$

which induce non-zero divisor of each germ $\mathcal{O}_{X,P}$ for all $P \in U_j$. We will say that $\{(U_j, f_j)\}_{j\in J}$ and $\{(U_j, f_j)\}_{j\in J}$ are equivalent if for every $j \in J$ there is invertible holomorphic function g_j on U_j such that $f_j = g_j h_j$. For any refinement $\{W_h\}_{h\in H}$ of $\{U_j\}_{j\in J}$ the family of pairs $\{(U_j, f_j)\}_{j\in J}$ induces a family of pairs $\{(W_h, \tilde{f}_h\}_{h\in H}$ on $\{W_h\}_{h\in H}$. Two families of pairs defined on two arbitrary open coverings, say $\{U_j\}_{j\in J}$ and $\{A_k\}_{k\in K}$, of X are said to be equivalent if there is a common open refinement $\mathfrak{V}$ of $\{U_j\}_{j\in J}$ and $\{A_k\}_{k\in K}$ such that the two families of pairs induced on $\mathfrak{V}$ are equivalent in the previous sense. An equivalence class of pairs will be called an effective Cartier divisor of X. A Cartier divisor of X is just the formal difference of two effective Cartier divisors. If each X_i is smooth, then we will often drop the word " Cartier ". In the next remark we will define and study the notions of closed analytic subset, closed irreducible analytic subset and decomposition into irreducuble components.

Remark 2. Let $Z \subseteq X$ be a closed subset. Hence Z is compact. We will say that Z is a closed analytic subset of X if there is an open covering $\mathfrak{U}$ of X such that for every $U \in \mathfrak{U}$ there is $\Lambda \subseteq H^0(U, \mathcal{O}_U)$ such that $Z \cap U = \{P \in U : f(P) = 0 \text{ for all } f \in \Lambda\}$. Since X is compact, we may always assume that $\mathfrak{U}$ is a finite covering. We may obviously refine $\mathfrak{U}$ to have the additional property that for every $U \in \mathfrak{U}$ there are a finite set $S_U \subseteq I$ and an open subset V of X_{S_U} such that $U = p_{S_U}^{-1}(V)$. Since every closed analytic subset of V is locally the zero-locus of finitely many holomorphic functions on open subsets of X_{S_U} and X is compact, we see that (refining if necessary again $\mathfrak{U}$) we may assume Λ finite for all $U \in \mathfrak{U}$. Since every germ of $\mathcal{O}_{X,P}$ depends only from finitely many variables, we get that for every closed analytic subset $Z \subseteq X$ there is a finite $S \subseteq I$ and a closed analytic $E \subseteq X_S$ such that $Z = E \times X_{I\setminus S}$. We will say that Z is irreducible if E is irreducible; since each X_i is assumed to be irreducible, the notion of irreducibility is well-defined, i.e. it does not depend from the choice of S. Similarly, for an arbitrary closed analytic subset $Z = E \times X_{I\setminus S}$ we will use a finite decomposition of E to define and prove the existence of the finite decomposition of Z into its irreducible components.

Now we define the notion of 1-form on X. We assume that each X_i is smooth. For any finite $S \subseteq I$ and any open subset U of $\prod_{i\in S} X_i$ set $\tilde{\Omega}^1_{X,fin}(U) := H^0(U, \Omega^1_U)$ and let $\Omega^1_{X,fin}$ be the $\mathcal{O}_X$-sheaf associated to this presheaf.

Remark 3. Let $z_1, \dots, z_n$ be the coodinates of $\mathbb{C}^n$ and $\Delta_n \subseteq \mathbb{C}^n$ any open polydisk. We have $\Omega^1_{\Delta_n} \cong \mathcal{O}^{\oplus n}_{\Delta_n}$ and every holomorphic 1-form on Δ_n may be written as $\omega = \sum_{i=1}^n h_i dz_i$ for suitable uniquely determined holomorphic functions $h_j : \Delta_n \to \mathbb{C}$. In particular every holomorphic 1-form on Δ_n may be written as a finite sum $\sum_j f_j d(g_j)$ of at most n terms with all f_j and all g_j holomorphic functions on Δ_n. Consider again our product X of compact complex manifolds and fix $P \in X$. For any integer $n \geq 1$ let $\Omega^1_{X,fin,P}[n]$ denotes the submodule of $\Omega^1_{X,fin,P}$ of all germs at P of 1-forms which may be written as the sum of at most n terms of the form $fd(g)$ with $f, g \in \mathcal{O}_{X,P}$. Thus $\Omega^1_{X,fin,P}[n] \subseteq \Omega^1_{X,fin,P}[n+1]$ for all $n \geq 1$ and we obtain an increasing filtration of $\Omega^1_{X,fin,P}$. The very definition of the product topology and of $\Omega^1_{X,fin,P}$ gives $\Omega^1_{X,fin,P} = \bigcup_{n \geq 1} \Omega^1_{X,fin,P}[n]$. Since X is compact, we have $\Omega^1_{X,fin}(X) = \bigcup_n \Omega^1_{X,fin}[n](X)$. It is easy to check that $\Omega^1_{X,fin}(U) = \bigcup_n \Omega^1_{X,fin}[n](U)$ for every open subset U of X of the form $V \times X_{I \setminus S}$ for some open $V \subseteq X_S$ whose cotangent bundle has a finite trivializing atlas and some finite $S \subseteq I$. Notice that the finiteness of a trivialing atlas is true for all V when each X_i is a compact Riemann surface.

Remark 4. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be products of smooth, compact and connected complex manifolds. Take any morphism $f : X \to Y$ as local ringed spaces. We define a pull-back map $f^* : \Omega^1_{Y,fin} \to \Omega^1_{X,fin}$ in the following way. Fix $P \in Y$ and $\omega \in \Omega^1_{Y,fin,P}$. By Remark 3 there is an integer n and germs $f_j, g_j \in \mathcal{O}_{Y,P}$ such that $\omega = \sum_{j=1}^n f_j d(g_j)$. Set $f^*(\omega) := \sum_{j=1}^n (f_j \circ f) d(g_j \circ f)$.

Remark 5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be products of smooth and compact complex manifolds. Take any morphism $f : X \to Y$ as local ringed spaces. By Remark 4 it is defined the pull-back map $f^* : \Omega^1_{Y,fin} \to \Omega^1_{X,fin}$. Any isomorphism between $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ induces an isomorphism between $\Omega^1_{Y,fin}$ and $\Omega^1_{X,fin}$.

Lemma 2. *Let* $X = \prod_{i \in I} X_i$ *be a product of smooth and compact complex manifolds and* $\omega \in H^0(X, \Omega^1_{X,fin})$. *Then there exist a finite* $S \subseteq I$ *and* $\eta \in H^0(X_S, \Omega^1_{X_S})$ *such that* $\omega = \pi_S^*(\eta)$.

Proof. By Remark 3 and the compactness of X there is an integer $n > 0$ and a finite open covering $\mathfrak{U}$ of X such that for every $U \in \mathfrak{U}$ there are $f_{U,j}, g_{U,j} \in H^0(U, \mathcal{O}_U)$, $1 \leq j \leq n$, such that $\omega | U = \sum_{j=1}^n f_{U,j} d(g_{U,j})$. Each $f_{U,j}$ and each $g_{U,j}$ depends only from the variables associated to a finite subset of I and the covering $\mathfrak{U}$ is finite, we are done. $\square$

Lemma 3. *Let* $X = \prod_{i \in I} X_i$ *be a product of smooth and compact complex manifolds and* $\omega \in H^0(X, \Omega^1_{X,fin})$. *Set* $Z := \{P \in X : \omega(P) = 0\}$. *Then* Z *is a closed analytic subset of* X *and there is a finite set* $S \subseteq X$ *and a closed analytic subset* E *of* X_S *such that* $Z = E \times X_{I \setminus S}$.

Proof. The first part follows from Lemma 2. The last assertion follows from the first part and Remark 2. $\square$

Lemma 4. *Let* $X = \prod_{i \in I} X_i$ *be a product of smooth and compact complex manifolds and* $\omega \in H^0(X, \Omega^1_{X,fin}) \setminus \{0\}$. *Assume the existence of an irreducible hypersurface* Y *of* X *contained in the zero-locus of* ω. *Then there exist an index* $i \in I$ *and an irreducible hypersurface* E *of* X_i *such that* $Y = E \times X_{I \setminus \{i\}}$.

Proof. By Lemma 2 it is sufficient to check to the case I finite. This case is proved in [1], Lemma at p. 248 and proof of Th. 2. $\square$

Remark 6. Let $X = \prod_{i\in I} X_i$ be a product of smooth and compact complex manifolds. As in the case of holomorphic 1-forms we may define the $\mathcal{O}_X$-sheaf $T_{X,fin}$ of locally finite vector fields and equip it with an increasing filtration $T_{X,fin}[n]$, $n \geq 1$, such that $T_{X,fin} = \bigcup_{n\geq 1} T_{X,fin}[n]$. With this definition Lemma 4 holds for $H^0(X, T_{X,fin})$.

Now we have some tools to attack the simplification problem in our category of products of compact complex spaces.

Remark 7. Fix any two finite-dimensional reduced compact complex spaces Y, Z and any two disjoint infinite sets E, F. Set $X := (\prod_{i\in E} X_i) \times (\prod_{i\in F} X_i)$ with $X_i := Y$ if $i \in E$ and $X_i := Z$ if $i \in F$. Obviously, $X \times Y \cong X \times Z$. A similar construction works (modulo set-theoretic axioms) if instead of two finite-dimensional reduced compact complex spaces Y, Z we take an arbitrary set of finite-dimensional complex spaces.

Remark 8. There are smooth elliptic curves E, F, G such that $E\times F \cong E\times G$, but $F \ncong G$ ([4]). Hence Theorem 1 is not true if we allow Z to have certain genus one components.

Remark 9. Of course, applying several times Theorem 1 we may get the same statement in which Z is a finite product of projective spaces and smooth compact Riemann surfaces of genus ≥ 2. By Remark 7 in the statement of Theorem 1 it is essential to assume that Z has only finitely many factors (even when X and Y have only finitely many factors).

Proof of Theorem 1. By Lemma 4 (case Z curve of genus $g \geq 2$) and Remark 6 (case $Z = \mathbf{P}^n$) we may repeat verbatim the proofs in [1] for the corresponding results when X and Y are finite-dimensional.

3. An almost metric of Kobayashi type

Notation 1. Set $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. We will always equip Δ with the Poincaré metric δ_Δ, i.e. with its Kobayashi metric ([3], p. 45).

Definition 1. Let X be a product of reduced and connected complex complex spaces and $P, Q \in X$. A chain γ connecting P and Q is a finite set of points of X, say P_i, $0 \leq i \leq k$, morphisms $h_i : \Delta \to X$, $1 \leq k$, and points $Q_i, Q_i' \in \Delta$ such that $P_0 = P$, $P_k = Q$, $h_i(Q_i) = P_{i-1}$ and $h_i(Q_i') = P_i$ for all i. Set $\|\gamma\| := \sum_{i=1}^{k} \delta_\Delta(Q_i, Q_i')$. If there is no such γ, then set $\delta_X(P, Q) = +\infty$. If there is at least one such γ, let $\delta_X(P, Q)$ be the infimum of the real numbers $\|\gamma\|$ for all γ. It is easy to check that $\delta_X(P, Q) \leq \delta_X(P, O) + \delta_X(O, Q)$ for all $P, Q, O \in X$. We will say that δ_X is the Kobayashi almost-distance of X.

Remark 10. Let X, A, B either a product of compact and connected reduced complex spaces or reduced and irreducible finite-dimensional complex spaces and $h : A \to X$, $f : X \to B$ morphisms. It is very easy to check that $\delta_B(f(P), f(Q)) \leq \delta_X(P, Q)$ and $\delta_X(h(O), h(R)) \leq \delta_A(O, R)$ for all $P, Q \in X$ and all $O, R \in A$.

4. PRODUCTS OF COMPACT RIEMANN SURFACES

From now on we assume that each X_i is a smooth, connected and one-dimensional, i.e. that each X_i is a compact and connected Riemann surface without boundary. We will call such X_I's products of Riemann surfaces. For any integer $j \geq 0$ set $I[j] := \{i \in I : p_a(X_i) = j\}$ and $X_{[j]} := X_{I[j]}$. Set $I[-] := I[0]$, $I[=] := I[1]$, $I[+] := \cup_{j \geq 2} I[j]$, $X^{[-]} := X_{I[-]}$, $X^{[=]} := X_{I[=]}$ and $X^{[+]} := X_{I[+]}$.

Remark 11. Fix integers $i, j \in I$ and a morphism $u : X_i \to X$. If there is no non-constant holomorphic map $X_i \to X_j$, then $\pi_j \circ u$ is constant. Hence $\pi_{[+]} \circ f(X^{[-]} \times X^{[=]})$ and $\pi_{[+] \cup [=]} \circ f(X^{[-]})$ are constant for every morphism $f : X \to X$.

From Remark 11 we get at once the following result.

Proposition 2. *Let X, Y be products of Riemann surfaces. If $X \cong Y$ (as local $\mathbb{C}$-ringed spaces), then $X^{[-]} \times X^{[=]} \cong Y^{[-]} \times Y^{[=]}$ and $X^{[-]} \cong Y^{[-]}$.*

Remark 12. Fix $P, Q \in X$, say $P = (P_i)_{i \in I}$ and $Q = (Q_i)_{i \in I}$. By Remark 10 we have $\delta_X(P, Q) \geq \delta_X(P_i, Q_i)$. If there is $i \in I[+]$ such that $P_i \neq Q_i$, then $\delta_X(P, Q) > 0$ (we allow the case $\delta_X(P, Q)$. Now assume $P_i = Q_i$ for all $i \in I[+]$. If there are only finitely many indices $i \in I$ such that $P_i \neq Q_i$, then both P, Q are contained in a finite-dimensional product T of curves of genus at most one and hence $\delta_X(P, Q)$ is finite and $\delta_X(P, Q) \leq \delta_T(P, Q) = 0$ (Remark 10). Since for fixed P the set of such Q's is dense in the slice $\{(P_i)_{i \in I[+]}\} \times X^{[+]}$, we get in this way an intrinsic characterization of $X^{[+]}$.

Theorem 2. *Let X, Y be products of Riemann surfaces. If $X \cong Y$ (as local $\mathbb{C}$-ringed spaces), then $X^{[-]} \cong Y^{[-]}$, $X^{[=]} \cong Y^{[=]}$ and $X^{[+]} \cong Y^{[+]}$.*

Proof. By Proposition 2 it is sufficient to prove $X^{[+]} \cong Y^{[+]}$. This is true by Remark 12. ☐

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Received: March 3, 2008