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A Characterization of Affine Varieties

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Abstract. Here we give a cohomological condition for the affiness of an algebraic scheme.

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1. Introduction

Recently, there was much new interest in cohomological characterizations of affine varieties and related problems (see [7], [8], [9]). Here we will piece together old stuff to get the following observation.

Theorem 1. *Let X be an algebraic scheme over an algebraically closed field $\mathbb{K}$. For any $P \in X$ write $Z(P) := \{Q \in X : f(Q) = f(P)$ for all $f \in H^0(X, \mathcal{O}_X)\}$. X is affine if and only each $Z(P)$ is finite and $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.*

Proof. since the " only if " part is a particular case of a classical criterion of Serre, we will only check the " if " part. By assumption for any integral curve $C \subset X$ there is $f \in H^0(X, \mathcal{O}_X)$ such that $f|C$ is not constant. X is quasi-affine ([3], Proposition 2). X is affine by [6], Th. 4.1. □

Remark 1. Let X be a reduced n-dimensional algebraic scheme over $\mathbb{K}$. Since $\dim(X) = n$, then $H^i(X, F) = 0$ for all $i > n$ and all coherent sheaves on X ([5], Th. III.2.7). If X has no complete n-dimensional irreducible component, then $H^n(X, F) = 0$ for every coherent sheaf F on X by a theorem of Lichtenbaum ([4], Theorem at p. 98). Hence if $Z(P)$ is finite for all $P \in X$, then $h^n(X, \mathcal{O}_X) = 0$ and in the statement of Theorem 1 we may just assume $h^i(X, F) = 0$ for $1 \le i \le n-1$. The analytic proof of [1], part a) of Teorema 1, works verbatim and gives that if $Z(P)$ is finite for all $P \in X$ and $\dim(H^{n-1}(X, \mathcal{O}_X)) < +\infty$, then $h^{n-1}(X, \mathcal{O}_X) = 0$. The analytic proof of [1], Proposizione 1, works verbatim and gives that if $Z(P)$ is

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finite for all $P \in X$, each $\mathcal{O}_{X,P}$, $P \in X$, is a Cohen-Macaulay local ring, and $\dim(H^i(X, \mathcal{O}_X)) < +\infty$ for all $1 \le i \le n-1$, then $h^i(X, \mathcal{O}_X) = 0$ for all i.

Remark 2. In the set-up of Theorem 1 assume $\mathbb{K} = \mathbb{C}$ and X normal. Let $(X_{an}, \mathcal{O}_{X_{an}})$ denote the complex-analytic space associated to $(X, \mathcal{O}_X)$. If $Z(P)$ is finite for all $P \in X$, then X is quasi-affine ([3], Proposition 2). Hence X is affine if and only if $(X_{an}, \mathcal{O}_{X_{an}})$ is Stein and $H^0(X, \mathcal{O}_X)$ is a finitely generated $\mathbb{C}$-algebra ([6], Prop. 5.5). Since the complex space associated to an affine variety is Stein, X_{an} is an open subset of a Stein complex space. Hence $(X_{an}, \mathcal{O}_{X_{an}})$ is Stein if and only if $H^i(X_{an}, \mathcal{O}_{X_{an}}) = 0$ for all $1 \le i \le \dim(X)-1$ ([2]).

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