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On Some Separation Axioms and Strongly Generalized Closed Sets in Bitopological Spaces

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Abstract

The aim of this paper is to introduce a new notion of a pairwise generalized closed set called a pairwise strongly generalized closed set and to study its basic properties. Furthermore, we introduce a new closure operator and study some sets properties of it by using this closure operator. We introduce four new classes of pairwise separation axioms. Also, we introduce ij-strongly g-continuous maps and ij-strongly g-irresolute maps.

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1 Introduction

The first step of generalized closed sets was done by Levine in 1970 [9]. He introduced the notion of $T_{1/2}$-spaces which is properly placed between T_0-space and T_1-space. Recently, Noiri [12] gave another generalization of Levine's g-closed set by utilizing the θ-closure operator. In 1986, Fukutake [3] generalized this notion to bitopological spaces and he defined a set A of a space X to be ij-generalized closed set (briefly ij-g-closed) if j-$cl(A) \subset U$ whenever $A \subset U$ and U is τ_i-open in X. Also, he defined a new closure operator and strongly pairwise $T_{1/2}$-space.

Some new types of generalized closed sets in bitopological spaces were introduced and studied in [2,6,7,8].

In this paper we generalize the notion of ij-strongly g-closed set which is stronger than the ij-generalized closed set introduced by Fukutake [3] and study its properties. Also, some lower pairwise separation axioms weaker than pairwise T_1 are investigated and their relationships with some other properties are studied. Furthermore, we define a new operation by using ij-strongly g-closed sets and give characterizations of a new topology τ_i^s and study some of their properties. Finally, we introduce ij-strongly g-continuous and ij-strongly g-irresoluteness by using these sets and study some of their basic properties and some of results for these functions.

Throughout this paper (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, ν_1, ν_2) (or briefly X, Y and Z) denote bitopological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, we shall denote the closure of A and the interior of A with respect to τ_i : (or σ_i) by i-$cl(A)$ and i-$\text{int}(A)$ respectively for $i = 1, 2$. Also $i, j = 1, 2$ and $i \neq j$. A point x of A is said to be in the ij-θ-interior of A, denoted by ij-$\text{int}_\theta(A)$ [1] if there exists $U \in \tau_i$ such that $x \in U \subset j$-$cl(U) \subset A$. A is said to be ij-θ-open [1] if $A = ij$-$\text{int}_\theta(A)$. The complement of an ij-θ-open is called ij-θ-closed (i.e. A is ij-θ-closed set if $A = ij$-$cl_\theta(A)$, where ij-$cl_\theta(A) = \{x \in X : j\text{-}cl(U) \cap A \neq \phi, U \in \tau_i$ and $x \in U\}$).

A subset A of X is said to be ij-regular open (resp. ij-regular closed) [13] if $A = i$-$\text{int}(j$-$cl(A))$ (resp. $A = i$-$cl(j$-$\text{int}(A)))$.

A subset A of X is said to be ij-semi-closed [2] (resp. ij-α-closed [4], ij-semi-pre-closed [5]) if j-$\text{int}(i$-$cl(A)) \subset A$ (resp. i-$cl(j$-$\text{int}(i$-$cl(A))) \subseteq A$, j-$\text{int}(i$-$cl(j$-$\text{int}(A))) \subset A$).

The complements of the above mentioned sets are called their respective open sets.

2 Preliminary Notes

Here we mention the following definitions and results

Definition 2.1 *A subset A of a bitopological space X is called:*

(i) ij-semi-generalized closed (briefly ij-sg-closed) [6] if ji-$scl(A) \subset U$ whenever $A \subset U$ and U is ij-semi-open in X.

(ii) ij-generalized semi closed (briefly ij-gs-closed) [6] if ji-$scl(A) \subset U$ whenever $A \subset U$ and U is τ_i-open in X.

(iii) ij-regular generalized closed (briefly ij-rg-closed) [7] if j-$cl(A) \subset U$ whenever $A \subset U$ and U is ij-regular open in X.

(iv) ij-generalized semi-pre-closed (briefly ij-gsp-closed) [7] if ji-$spcl(A) \subset U$ whenever $A \subset U$ and U is τ_i-open in X.

v ij-generalized α-closed (briefly ij-$g\alpha$-closed) [6] if ji-$\alpha cl(A) \subset U$ whenever $A \subset U$ and U is ij-α-open in X.

(vi) ij-α-generalized closed (briefly ij-αg-closed) [7] ji-$\alpha cl(A) \subset U$ whenever $A \subset U$ and U is τ_i-open in X.

(vii) ij-θ-generalized closed (briefly ij-θg-closed) [7] if ji-$cl_\theta(A) \subset U$ whenever $A \subset U$ and U is τ_i-open in X.

The complement of the above-mentioned sets are called their respective open sets.

Definition 2.2 *A bitopological space* (X, τ_1, τ_2) *is said to be:*

(i) pairwise T_0[10] if for every pair of points x and y such that $x \neq y$ there exists a τ_1-open set containing x but not y or a τ_2-open set containing y but not x.

(ii) pairwise T_1[10] if for every pair of points x and y such that $x \neq y$, there exist a τ_1-open set and τ_2-open set such that $x \in U, y \notin U$, $y \in V$ and $x \notin V$.

(iii) ij-$T_{1/2}$-space [3] if every ij-g-closed set in X is τ_j-closed in X.

Definition 2.3 [6]. *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called*

(i) ij-generalized continuous (briefly ij-g-continuous) if $f^{-1}(V)$ is ij-g-closed in X for every σ_j-closed set V in Y.

(ii) ij-generalized semi-continuous (briefly ij-gs-continuous) if $f^{-1}(V)$ is ij-gs-closed set in X for every σ_j-closed set V of Y.

(iii) ij-generalized-irresolute (briefly ij-g-irreslute) if $f^{-1}(V)$ is ij-g-closed in X for every ij-g-closed set V in Y.

Definition 2.4 [3]. *For any subset* A *of* X, *define* ij-$gcl(A) = \cap\{U : A \subseteq U$ *and* U *is* ij-g-*closed*$\}$ *and* $\tau_i^* = \{F : ij\text{-}gcl(F^c) = F^c\}$.

Lemma 2.5 [3]. *A bitopological space is pairwise $T_{1/2}$ if and only if $\{x\}$ is τ_j-open or τ_i-closed for each $x \in X$.*

Lemma 2.6 [11]. *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an i-closed, then for each subset $S \subset Y$ and each τ_i-open set U containing $f^{-1}(S)$, there is a σ_i-open set V containing S such that $f^{-1}(V) \subset U$.*

Lemma 2.7 *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is surjective j-continuous, then for every subset B of Y, $j\text{-}cl(f^{-1}(B)) \subset f^{-1}(j\text{-}cl(B))$.*

Proof. Let $x \in j\text{-}cl(f^{-1}(B))$. Suppose that V be j-open set of Y containing $f(x)$, i.e. $f(x) \in V$. Then $x \in f^{-1}(V)$. Since $f^{-1}(V)$ is j-open of X, then $f^{-1}(V) \cap f^{-1}(B) \neq \phi$ this implies that $f^{-1}(V \cap B) \neq \phi$ and $V \cap B \neq \phi$. Thus $f(x) \in j\text{-}cl(B)$ and $x \in f^{-1}(f(x)) \in f^{-1}(j\text{-}cl(B))$. This means that $x \in f^{-1}(j\text{-}cl(B))$. Hence $j\text{-}cl(f^{-1}(B)) \subset f^{-1}(j\text{-}cl(B))$.

Lemma 2.8 *A bijection $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is i-open if and only if f is i-closed.*

Proof. Let F be i-closed set of X. Then $F = X \setminus U$ where U is i-open set. Hence $f(F) = f(X \setminus U) = Y \setminus f(U)$. Since f is i-open, then $f(U)$ is i-open and $Y \setminus f(U)$ is i-closed in Y. Thus $f(F)$ is i-closed and f is i-closed. The proof of the converse is similar.

3 Basic properties of pairwise strongly generalized closed sets

Definition 3.1 *A subset A of a space X is called ij-strongly generalized closed (briefly ij-g^s-closed) set if $j\text{-}cl(A) \subset U$ whenever $A \subset U$ and U is ij-g-open in X. If $A \subset X$ is 12-g^s-closed and 21-g^s-closed, then it said to be pairwise strongly generalized closed (briefly P-g^s-closed).*

Theorem 3.2 *Every j-closed set in X is ij-g^s-closed in X but not conversely.*

Proof. *Let $A \subset X$ be j-closed and $A \subset U$ such that U is ij-g-open. Then $j\text{-}cl(A) = A$ and so $j\text{-}cl(A) \subset U$. Thus A is ij-g^s-closed.*

The converse of the above theorem need not be true as is seen from the following example.

Example 3.3 *Let $X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{c\}, \{a, c\}, X\}$. The set $A = \{b, c\}$ is 12-g^s-closed but A not 2-closed, since 2-$cl(A) = X$.*

Theorem 3.4 *Every ij-g^s-closed set in X is an ij-g-closed set in X but not conversely.*

Proof. Let A be an ij-g^s-closed set. Let $A \subseteq U$, where U is P-open. Since every P-open set is ij-g-open and A is ij-g^s-closed, we have j-$cl(A) \subset U$. Therefore A is ij-g-closed

The converse need not be true as is seen from the following example.

Example 3.5 *Consider the space $X = \{a, b, c\}$ with $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, X\}$. The set $\{b\}$ is 12-g-closed but not 12-g^s-closed, since* 2-$cl(\{b\}) = X \not\subset \{a, b\}$ *where $\{a, b\}$ is 12-g-open.*

Remark 3.6 *Every ij-g^s-closed set in X is ij-αg-closed, ij-gs-closed, ij-gsp-closed and ij-rg-closed but not conversely as is seen from Example 3.5.*

Remark 3.7 *The concept of ij-g^s-closed set is independent of the following class of sets namely ji-α-closed set, ij-gα-closed set, ji-semi-closed set, ij-sg-closed set and ij-θg-closed.*

Example 3.8 *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, X\}$. In this space, the set $\{b\}$ is 21-α-closed, 12-gα-closed, 21-semi-closed and 12-sg-closed but not 12-g^s-closed, since 2-$cl(\{b\}) = \{b, c\} \not\subset \{a, b\}$ where $\{a, b\}$ is 12-g-open. Also, the set $\{a, c\}$ is 12-g^s-closed but not any of the sets mentioned above.*

From the above discussion and from the results in [7], we have the following diagram.

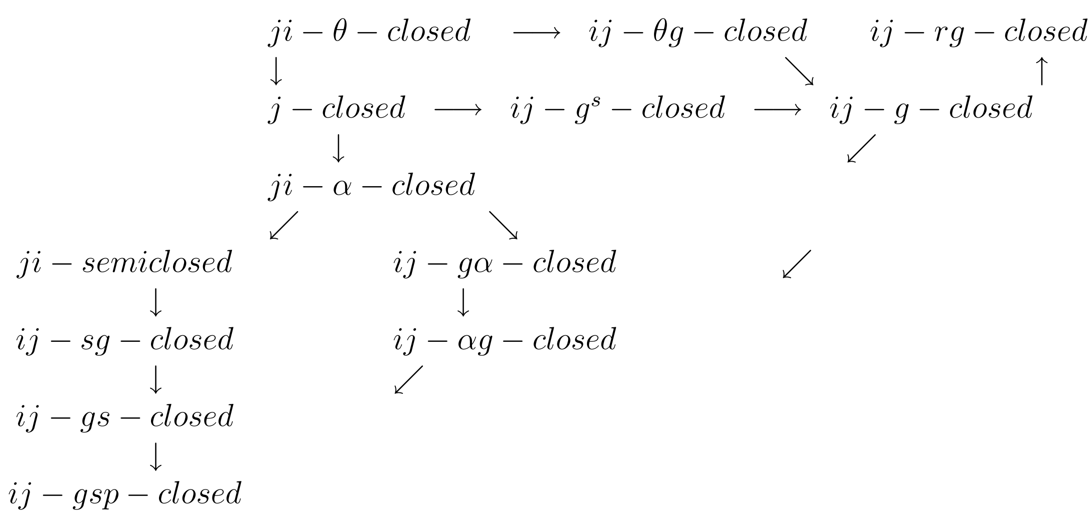

Figure (1)

Theorem 3.9 *Union of two ij-g^s-closed sets in X is ij-g^s-closed in X.*

Proof. *Let $A, B \subset X$ be ij-g^s-closed sets. Let U be an ij-g-open subset of X such that $A \cup B \subset U$. We have j-$cl(A \cup B) = j$-$cl(A) \cup j$-$cl(B) \subset U \cup U = U$, since A and B are ij-g^s-closed. Hence $A \cup B$ is ij-g^s-closed.*

Theorem 3.10 *A subset A of X is ij-g^s-closed if and only if j-$cl(A) \setminus A$ contains no nonempty ij-g-closed set in X.*

Proof. *Let F be an ij-g-closed subset of j-$cl(A) \setminus A$. Now, $F \subset j$-$cl(A) \setminus A$ and $A \subset X \setminus F$ where A is ij-g^s-closed and $X \setminus F$ is ij-g-open. Thus j-$cl(A) \subset X \setminus F$ or equivalently $F \subset X \setminus j$-$cl(A)$. By assumption we have that $F \subset j$-$cl(A)$, and so $F \subset (X \setminus j$-$cl(A) \cap j$-$cl(A)) = \phi$. This show that F is empty.*

Conversely assume j-$cl(A) \setminus A$ contains no nonempty ij-g-closed set. Let $A \subseteq U, U$ is ij-g-open. Suppose that j-$cl(A)$ is not contained in U. Then j-$cl(A) \cap U^c$ is a nonempty ij-g-closed set of j-$cl(A) \setminus A$, which is a contradiction. Therefore, j-$cl(A) \subset U$ and hence A is ij-g^s-closed.

Theorem 3.11 *If a subset A of X is ij-g^s-closed and $A \subseteq B \subseteq j$-$cl(A)$, then B is ij-g^s-closed in X.*

Proof. *Let $B \subset U$, where U is ij-g-open. Since A is ij-g^s-closed and $A \subset B$ it follows that $A \subset U$. By hypothesis $B \subset j$-$cl(A)$ and hence j-$cl(B) \subset j$-$cl(A) \subset U$. Thus B is ij-g^s-closed.*

Definition 3.12 *A subset A of a bitopological space X is called ij-strongly generalized open set (briefly ij-g^s-open) if A^c is ij-g^s-closed.*

Theorem 3.13 *A subset A of X is ij-g^s-open in X if and only if $F \subset j$-$int(A)$ whenever $F \subseteq A$ and F is ij-g-closed in X.*

Proof. *Let A be ij-g^s-open and suppose $F \subset A$ where F is ij-g-closed. Then $X \setminus A$ is ij-g^s-closed and $X \setminus A \subset X \setminus F$, where $X \setminus F$ is ij-g-open set. This implies that j-$cl(X \setminus A) \subset X \setminus F$. Now j-$cl(X \setminus A) = X \setminus j$-$int(A)$. Hence $X \setminus j$-$int(A) \subset X \setminus F$ and $F \subset j$-$int(A)$.*

Conversely, if F is an ij-g-closed set with $F \subset j$-$int(A)$ whenever $F \subset A$. Then $X \setminus A \subset X \setminus F$ and $X \setminus j$-$int(A) \subset X \setminus F$. Thus j-$cl(X \setminus A) \subset X \setminus F$. Hence $X \setminus A$ is ij-g^s-closed and A is ij-g^s-open.

Theorem 3.14 *For each $x \in X, \{x\}$ is ij-g-closed in X or $\{x\}^c$ is ij-g^s-closed in X.*

Proof. *If $\{x\}$ is not ij-g-closed, then the only j-g-open set containing $\{x\}^c$ is X. Thus j-$cl(\{x\}^c) \subset X$ and $\{x\}^c$ is ij-g^s-closed.*

Lemma 3.15 *If A and B are two ij-g^s-open subset of a bitopological space X, then $A \cap B$ is ij-g^s-open.*

Proof. *Suppose that F is ij-g-closed set contained in $A \cap B$. Since A and B are ij-g^s-open sets, then by Theorem 3.13, $F \subset j$-int(A) and $F \subset j$-int(B). Thus $F \subset j$-int$(A) \cap j$-int$(B) = j$-int$(A \cap B)$. Hence $F \subset j$-int$(A \cap B)$ and therefore $A \cap B$ is ij-g^s-open.*

4 Applications of ij-g^s-closed sets

In this section we introduce four new classes of bitopological spaces called T_b, T_d, T_p and T_s.

Definition 4.1 *A bitopological space X is called ij-T_b-space if every ij-gs-closed set is j-closed in X.*

Definition 4.2 *A bitopological space X is called ij-T_d-space if every ij-gs-closed set is ij-g-closed.*

Remark 4.3 *P-T_1 and P-T_b spaces are independent as is seen from the following example.*

Example 4.4 *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi,\{a\},\{a, b\}, X\}$ and $\tau_2 = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. Then (X, τ_1, τ_2) is 12-T_1 but not 12-T_b, since the set $A = \{b\}$ is 12-gs-closed but it is not 2-closed. Also, if σ_1 is indiscrete topology and σ_2 is discrete topology, then (X, σ_1, σ_2) is 12-T_b but not 12-T_1.*

Remark 4.5 *P-T_0 and P-T_d spaces are independent as is seen from the following example*

Example 4.6 *Let $X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\tau_2 = \{\phi, \{b\}, X\}$. Then (X, τ_1, τ_2) is 12-T_0 but not 12-T_d, since the set $\{a\}$ is 12-gs-closed but it is not 12-g-closed, since 2-cl$(\{a\}) = \{a, c\} \not\subset \{a\} \in \tau_1$. Also, let $X = \{a, b, c\}$, $\sigma_1 = \{\phi, \{a\}, X\}$ and $\sigma_2 = \{\phi, \{b, c\}, X\}$. Then (X, σ_1, σ_2) is 12-T_d but not 12-T_0.*

Definition 4.7 *A bitopological space X is called ij-T_p-space if every ij-g^s-closed set is j-closed in X.*

Definition 4.8 *A bitopological space X is called ij-T_s-space if every ij-g-closed set is ij-g^s-closed in X.*

Theorem 4.9 *If X is ij-$T_{1/2}$, then it is ij-T_p but not conversely.*

Proof. *Let X be a P-$T_{1/2}$-space. Since every ij-g^s-closed set is ij-g-closed and X is P-$T_{1/2}$, then X is P-T_p.*

The converse need not be true. To see this, we have the following example.

Example 4.10 *Let $X = \{a, b, c\}$ with $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{b, c\}, X\}$. Let $A = \{b\}$. Then A is 12-g-closed but A not 2-closed. Hence (X, τ_1, τ_2) is 12-T_p-space but not 12-$T_{1/2}$.*

Remark 4.11 *The spaces P-T_0 and P-T_p are independent as is seen from the following example.*

Example 4.12 *Let $X = \{a, b, c\}$,$\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$,$\tau_2 = \{\phi, \{c\}, \{a, c\}, X\}$, $\sigma_1 = \{\phi, \{a\}, X\}$ and $\sigma_2 = \{\phi, \{b, c\}, X\}$. Then the space (X, τ_1, τ_2) is 12-T_0 but not 12-T_p and the space (X, σ_1, σ_2) is 12-T_p space but not 12-T_0-space.*

Theorem 4.13 *If X is P-$T_{1/2}$, then it is P-T_s but not conversely.*

Proof. *Let X be a P-$T_{1/2}$ space. Let A be ij-g-closed set in P-T_s-space. Since X is P-$T_{1/2}$, then A is j-closed in X. Since every j-closed set is ij-g^s-closed, A is ij-g^s-closed. Hence X is P-T_s.*

The converse need not be true as is seen from the following example.

Example 4.14 *Let $X = \{a, b, c\}, \tau_1 = \{\phi, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b, c\}, X\}$. Then X is 12-T_s but not 12-$T_{1/2}$, since $\{c\}$ is 12-g-closed but 2-$cl(\{c\}) = \{b, c\}$, i.e, $\{c\}$ is not 2-closed.*

Remark 4.15 *P-T_s and P-T_p spaces are independent as is seen from the following example.*

Example 4.16 *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\phi, \{b, c\}, X\}$. Then (X, τ_1, τ_2) is 12-T_p but not 12-T_s, since $\{a, c\}$ is 12-g-closed but not 12-g^s-closed. Also, if $\sigma_1 = \{\phi, \{a, b\}, X\}, \sigma_2 = \{\phi, \{a\}, \{b, c\}, X\}$. Then (X, σ_1, σ_2) is 12-T_s but not 12-T_p, since the set $\{c\}$ is 12-g^s-closed but not 2-closed.*

Remark 4.17 *A P-T_s space need not be P-T_0 as is seen from the following example.*

Example 4.18 *Let $X = \{a, b, c\}, \tau_1 = \{\phi, \{a, b\}, X\}$ and $\tau_2 = \{\phi, X\}$. Then X is 12-T_s but not 12-T_0.*

Theorem 4.19 *If X is P-T_b space, then it is P-T_p but not conversely*

Proof. *Let X be a P-T_b space, and let A be 12-g^s-closed set in X. Then A is 12-gs-closed in X. Since X is P-T_b, A is j-closed in X. Hence X is a P-T_p space.*

The converse need not be true as is seen from the following example .

Example 4.20 *Let $X = \{a, b, c\}$ with $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{b, c\}, X\}$. Then X is 12-T_p but not 12-T_b, since the set $\{b\}$ is 12-gs-closed but $\{b\}$ is not 2-closed.*

Remark 4.21 *P-T_s and P-T_d spaces are independent as is see from the following example*

Example 4.22 *Let $X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{b, c\}, X\}$. Then (X, τ_1, τ_2) is 12-T_d but not 12-T_s. Also, let $\sigma_1 = \{\phi, \{a, b\}, X\}, \sigma_2 = \{\phi, \{a\}, \{b, c\}, X\}$. Then (X, σ_1, σ_2) is 12-T_s but not 12-T_d, since the set $\{b\}$ is 12-gs-closed but not 12-g-closed.*

Theorem 4.23 *If X is a P-T_b space, then it is P-T_s but not conversely.*

Proof. *Let X be a P-T_b space and A be any ij-g-closed set in X. Then A is ij-gs-closed in X. Since X is P-T_b, A is j-closed in X and hence ij-g^s-closed in X. Therefore X is P-T_s.*

The converse of above theorem need not be true as is seen from the following example.

Example 4.24 *Let $X = \{a, b, c\}$, with $\tau_1 = \{\phi, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b, c\}, X\}$. Then X is 12-T_s but not 12-T_b, since $\{c\}$ is 12-gs-closed set but $\{c\}$ is not 2-closed*

Remark 4.25 *P-T_d and P-T_p spaces are independent as is seen from the following example*

Example 4.26 *Let $X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{b, c\}, X\}$. Then (X, τ_1, τ_2) is 12-T_d but not 12-T_p, since the set $\{a, c\}$ is 12-g^s-closed but not 2-closed. Also, let $\sigma_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then (X, σ_1, σ_2) is 12-T_p but not 12-T_d, since $\{b\}$ is 12-gs-closed but not 12-g-closed where 2-cl$(\{b\}) = \{b, c\} \not\subset \{b\}$.*

Theorem 4.27 *X is P-$T_{1/2}$ if and only if X is both P-T_p and P-T_s.*

Proof. *From Theorem 4.9 and 4.13, necessity follows. For the sufficiency, let A be any ij-g-closed set in X. Since X is P-T_s A is ij-g^s-closed. Since X is P-T_p, A is j-closed in X. Therefore X is P-$T_{1/2}$*

Theorem 4.28 *A space X is P-T_p if and only if for each $x \in X, \{x\}$ is ij-g-closed or j-open.*

Proof. *Suppose that X is P-T_p and for each $x \in X, \{x\}$ is not ij-g-closed. Since X is the only ij-g-open set containing $\{x\}^c, \{x\}^c$ is ij-g^s-closed and thus j-closed. Hence $\{x\}$ is j-open.*

To prove the converse assume that A is ij-g^s-closed set in X. Let $x \in$ j-cl(A), then we have two case.
case (i) If $\{x\}$ is ij-g-closed and if $x \notin A$, then $\{x\} \subseteq$ j-cl$(A) \setminus A$. Thus contradicts Theorem 3.10 Therefor $x \in A$.
case (ii) If $\{x\}$ is j-open, then $\{x\} \cap A \neq \phi$ and so $x \in A$.
In either case, $x \in A$ and hence A is j-closed.

Theorem 4.29 *If X is P-T_p with $Y \subseteq X$, then Y is P-T_p*

Proof. *For $y \in Y, \{y\}$ is j-open or ij-g-closed in X. Using Theorem 4.28, $\{y\}$ is j-open or -ij-g-closed in Y.*

Remark 4.30 *From the above discussion and examples we have the following diagram*

$$\begin{array}{ccccc} T_1 & \longrightarrow & T_{1/2} & \longrightarrow & T_0 \\ & \swarrow & & \searrow & \\ T_p & \longrightarrow & T_b & \longrightarrow & T_s \end{array}$$

Figure (2)

5 Characterization of ij-strongly generalized closure operator

In this section, we introduce a new closure operator and a new bitopology using this closure operator and study some of their properties.

Definition 5.1 *Let (X, τ_1, τ_2) be a bitopological space and $E \subset X$. We define the ij-g^s-closure operator of E and denoted by ij-$g^s cl(E)$ as, ij-$g^s cl(E) = \cap\{A : E \subseteq A$ and A is ij-g^s-closed set$\}$.*

Lemma 5.2 *If $E \subseteq X$, then $E \subseteq ij$-$gcl(E) \subseteq ij$-$g^s cl(E) \subseteq j$-$cl(E)$*

Proof. *Since every j-closed set is ij-g^s-closed and since every ij-g^s-closed set is ij-g-closed, the proof follows.*

Remark 5.3 *The containment relations in the Lemma 5.2 may be proper. consider the topological space $X = \{a, b, c\}$ with $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi,\{c\},\{a, c\},X\}$. Since 12-$g^s cl(\{c\}) = \{b, c\}$ and 2-$cl(\{c\}) = X$, we have $\{c\} \subset 12$-$g^s cl(\{c\}) \subset 2$-$cl(\{c\})$. Also, in the space $X = \{a, b, c\}$ with $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b, c\}, X\}$, we have 12-$gcl(\{c\}) = \{c\}$ and 12-$g^s cl(\{c\}) = \{b, c\}$. Therefore, $\{c\} \subset 12$-$gcl(\{c\}) \subset 12$-$g^s cl(\{c\})$.*

Theorem 5.4 *ij-$g^s cl$ is a Kuratowski closure operator on X.*

Proof. (i) ij-$g^s cl(\phi) = \phi$ and $E \subseteq ij$-$g^s cl(E)$ follows from Lemma 5.2.

(ii) To prove ij-$g^s cl(E_1) \cup ij$-$g^s cl(E_2) \subseteq ij$-$g^s cl(E_1 \cup E_2)$ we have ij-$g^s cl(E_1) \subset ij$-$g^s cl(E_1 \cup E_2), i = 1, 2$. Therefore, ij-$g^s cl(E_1) \cup ij$-$g^s cl(E_2) \subseteq ij$-$g^s cl(E_1 \cup E_2)$. To prove: ij-$g^s cl(E_1 \cup E_2) \subseteq ij$-$g^s cl(E_1) \cup ij$-$g^s cl(E_2)$, let x be any point such that $x \notin ij$-$g^s cl(E_1) \cup ij$-$g^s cl(E_2)$. Since $x \notin ij$-$g^s cl(E_1)$ and $x \notin ij$-$g^s cl(E_2)$, there exists two ij-g^s-closed sets F_1 and F_2 such that $E_1 \subseteq F_1$ and $E_2 \subseteq F_2, x \notin F_1$ and $x \notin F_2$. Then

$x \notin F_1 \cup F_2, E_1 \cup E_2 \subseteq F_1 \cup F_2$ and $F_1 \cup F_2$ is ij-g^s-closed by Theorem 3.9. Thus we have $x \notin ij\text{-}g^s cl(E_1 \cup E_2)$. Therefore, we have ij-$g^s cl(E_1 \cup E_2) \subseteq ij\text{-}g^s cl(E_1) \cup ij\text{-}g^s cl(E_2)$. Hence $ij\text{-}g^s cl(E_1 \cup E_2) = ij\text{-}g^s cl(E_1) \cup ij\text{-}g^s cl(E_2)$.

(iii) Let A be any ij-g^s-closed set in X containing E. Then by definition ij-$g^s cl(E) \subseteq A$. Since A is ij-g^s-closed and contains $ij\text{-}g^s cl(E)$, $ij\text{-}g^s cl(ij\text{-}g^s cl(E)) \subseteq A$. This means that $ij\text{-}g^s cl(ij\text{-}g^s cl((E))$ is contained in every ij-g^s-closed set containing E. Hence $ij\text{-}g^s cl(ij\text{-}g^s cl(E)) \subseteq ij\text{-}g^s cl(E)$. Therefore, $ij\text{-}g^s cl(ij\text{-}g^s cl(E)) = ij\text{-}g^s cl(E)$.

Definition 5.5 *Let τ_i^s be the topology in X generated by ij-$g^s cl$ in the usual manner. That is $\tau_i^s = \{F : ij\text{-}g^s cl(F^c) = F^c\}$.*

Theorem 5.6 *For any space (X, τ_1, τ_2), the following hold:*

(i) $\tau_j \subseteq \tau_i^s \subseteq \tau_i^*$

(ii) The space (X, τ_1, τ_2) is P-T_p if and only if $\tau_j = \tau_i^s$.

Proof. To prove $\tau_j \subseteq \tau_i^s$, let $E \in \tau_j$ and E^c is j-closed. By Lemma 5.2, we have $E^c \subseteq ij\text{-}g^s cl(E^c) \subseteq j\text{-}cl(E^c) = E^c$. This implies that, $ij\text{-}g^s cl(E^c) = E^c$ and hence $E \in \tau_i^s$. To prove $\tau_i^s \subseteq \tau_i^*$, let $E \in \tau_i^s$, then $ij\text{-}g^s cl(E^c) = E^c$. By using Lemma 5.2, $E^c \subseteq ij\text{-}gcl(E^c) \subseteq ij\text{-}g^s cl(E^c) = E^c$. Thus $ij\text{-}gcl(E^c) = E^c$ and hence $ij\text{-}g^s cl(E^c) = E^c$. Therefore $E \in \tau_i^*$ and $\tau_i \subseteq \tau_i^s \subseteq \tau_i^*$

(ii) Assume that $\tau_j = \tau_i^s$ and let $A \subseteq X$ be ij-g^s-closed in (X, τ_1, τ_2). Then $A = ij\text{-}g^s cl(A)$ and so $A^c \in \tau_i^s$, this implies that $A^c \in \tau_j = \tau_i^s$. Thus (X, τ_1, τ_2) is P-T_p. Conversely, assume that (X, τ_1, τ_2) is P-T_p. Then every ij-g^s-closed set is j-closed in X. Therefore $\tau_i^s \subseteq \tau_j$ and hence $\tau_i = \tau_i^s$.

Theorem 5.7 *If a space (X, τ_1, τ_2) is P-$T_{1/2}$, then (X, τ_1^s, τ_2^s) is P-$T_{1/2}$.*

Proof. *Let $x \in X$ be any point. By Theorem 3.14, $\{x\}$ is ij-g-closed in (X, τ_1, τ_2) or $\{x\}^c$ is ij-g^s-closed in (X, τ_1, τ_2), If $\{x\}$ is ij-g-closed, then $\{x\}$ is τ_j-closed and hence $\{x\}$ is τ_j^s-closed. If $\{x\}^c$ is ij-g^s-closed in (X, τ_1, τ_2), by the definition of $\tau_i^s, \{x\}$ is j-open in τ_i^s. Therefore (X, τ_1^s, τ_2^s) is P-$T_{1/2}$ by Lemma 2.5*

Corollary 5.8 *For a P-$T_{1/2}$-space, $(X, \tau_1, \tau_2), (\tau_i^s)^s = \tau_i^s \quad (i = 1, 2)$,*

Proof. *By Theorem 5.7, (X, τ_1^s, τ_2^s) is ij-$T_{1/2}$. But every ij-$T_{1/2}$-space is ij-T_p. Thus by Theorem 5.6 we have $(\tau_i^s)^s = \tau_i^s$.*

Theorem 5.9 *Let (X, τ_1, τ_2) be a bitopological space. Then the following statements are equivalent.*

(i) (X, τ_1^s, τ_2^s) *is discrete*

(ii) For each $x \in X, \{x\}^c$ *is* ij*-*g^s*-closed in* (X, τ_1, τ_2).

(iii) If $\{x\}$ *is* ij*-g-closed in* $(X, \tau_1, \tau_2), \{x\}$ *is* j*-open in* (X, τ_1, τ_2).

Proof. *(i)⇒(ii) If* (X, τ_1^s, τ_2^s) *is discrete, then for each* $x, \{x\}^c = ij$*-*$g^s cl(\{x\}^c) = \cap\{A : \{x\}^c \subseteq A$ *and* A *is* ij*-*g^s*-closed set in* $(X, \tau_1^s, \tau_2^s)\}$*, it follows that* $\{x\}^c$ *is itself* ij*-*g^s*-closed in* (X, τ_1, τ_2).

(ii)⇒(i) Suppose $\{x\}$ *is* ij*-g-closed in* (X, τ_1, τ_2)*. Then* $\{x\}^c$ *is* ij*-g-open and* ij*-*g^s*-closed in* (X, τ_1, τ_2) *by (ii). Thus we have* j*-*$cl(\{x\}^c) \subseteq \{x\}^c$*. Therefore,* $\{x\}$ *is* τ_j*-open in* (X, τ_1, τ_2).

(iii)⇒(i) Let $x \in X$ *be any point. If* $\{x\}$ *is* ij*-g-closed in* (X, τ_1, τ_2)*, then by using (iii) and Theorem 5.6,* $\{x\}$ *is* τ_j*-open and thus* $\{x\} \in \tau_i^s$*. If* $\{x\}$ *is not* ij*-g-closed in* (X, τ_1, τ_2)*, then by using Theorem 3.14,* $\{x\}^c$ *is* ij*-*g^s*-closed in* (X, τ_1, τ_2)*. This implies* j*-*$cl(\{x\}^c) \subseteq \{x\}^c$*. Thus* $\{x\} \in \tau_j$ *and hence* $\{x\} \in \tau_i^s$.

6 On ij-g^s-continuous and ij-g^s-irresolute functions

Definition 6.1 *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called:*

(i) ij-strongly g-continuous (briefly ij-g^s-continuous) if $f^{-1}(V)$ is ij-g^s-closed in X for every σ_j-closed set V of Y.

(ii) ij-strongly g-irresolute (briefly ij-g^s-irresolute) if $f^{-1}(V)$ is ij-g^s-closed in X for every ij-g^s-closed set V of Y.

Theorem 6.2 *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a function:*

(i) f *is* ij*-*g^s*-continuous.*

(ii) For each $x \in X$ *and for each* σ_j*-open set* V *containing* $f(x)$*, there is an* ij*-*g^s*-open set* U *containing* x *such that* $f(U) \subset V$.

(iii) $f(ij$*-*$g^s cl(A)) \subset j$*-*$cl(f(A))$ *for each subset* A *of* X.

(iv) ij*-*$g^s cl(f^{-1}(B)) \subset f^{-1}(j$*-*$cl(B))$ *for each subset* B *of* Y.

Then (i)⇒(ii)⇒(iii)⇒(iv).

Proof. *(i)⇒(ii): Let* $x \in X$ *and* V *be* σ_j*-open set containing* $f(x)$*. Then by (i),* $f^{-1}(V)$ *is* ij*-*g^s*-open set of* X *which containing* x*. If* $U = f^{-1}(V)$*, then* $f(U) \subset V$.

(ii)⇒(iii): Let A *be a subset of a space* X *and* $f(x) \notin j$*-*$cl(f(A))$*, then there exists* σ_j*-open set* V *of* Y *containing* $f(x)$ *such that* $V \cap f(A) = \phi$*. Then by (ii),*

there is an ij-g^s*-open set* U *such that* $f(x) \in f(U) \subset V$. *Hence* $f(U) \cap f(A) = \phi$ *and* $U \cap A = \phi$. *Consequently,* $x \notin ij$-$g^s cl(A)$ *and* $f(x) \notin f(ij$-$g^s cl(A))$.

(iii)⇒(iv): Let B *be a subset of* Y *and* $A = f^{-1}(B)$. *By (iii),* $f(ij$-$g^s cl(f^{-1}(B))) \subset j$-$cl(f(f^{-1}(B))) \subset j$-$cl(B)$. *Thus* ij-$g^s cl(f^{-1}(B)) \subset f^{-1}(j$-$cl(B))$.

Theorem 6.3 *If a function* $f : X \to Y$ *is* j*-continuous, then* f *is* ij-g^s*-continuous.*

Proof. *Let* V *be a* σ_j*-closed set of* Y. *Since* f *is* j*-continuous, then* $f^{-1}(V)$ *is* τ_j*-closed. By Theorem 3.2,* $f^{-1}(V)$ *is* ij-g^s*-closed and hence* f *is* ij-g^s*-continuous.*

The converse of the above theorem is not true in general and this can easily be seen from the following example

Example 6.4 *Let* $X = Y = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\phi, \{c\}, \{a, c\}, X\}$, $\sigma_1 = \{\phi, \{a, b\}, Y\}$ *and* $\sigma_2 = \{\phi, \{a\}, \{c\}, \{a, c\}, Y\}$. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be the identity function. Clearly,* f *is 12-*g^s*-continuous but* f *is not 2-closed.*

Theorem 6.5 *If a function* $f : X \to Y$ *is* ij-g^s*-continuous, then* f *is* ij-g*-continuous.*

Proof. *Let* V *be a* σ_j*-closed set of* Y. *Since* f *is* ij-g^s*-continuous, then* $f^{-1}(V)$ *is* ij-g^s*-closed. By Theorem 3.4,* $f^{-1}(V)$ *is* ij-g*-closed and hence* f *is* ij-g*-continuous.*

The following example show that an ij-g*-continuous function need not be* ij-g^s*-continuous.*

Example 6.6 *Let* $X = \{a, b, c\}, \tau_1 = \{\phi, \{c\}, \{b, c\}, X\}, \tau_2 = \{\phi, \{a, b\}, X\}$, $\sigma_1 = \{\phi, \{a\}, X\}$ *and* $\sigma_2 = \{\phi, \{b\}, \{b, c\}, X\}$. *Let* $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ *be identity function. Clearly* f *is 12-*g*-continuous but* f *is not 12-*g^s*-continuous.*

From above and by [6] also, it is clear that ij-g^s*-continuous and* ij-gs*-continuous are independent concepts. Thus we have the following implication and none of them is be reversible:*

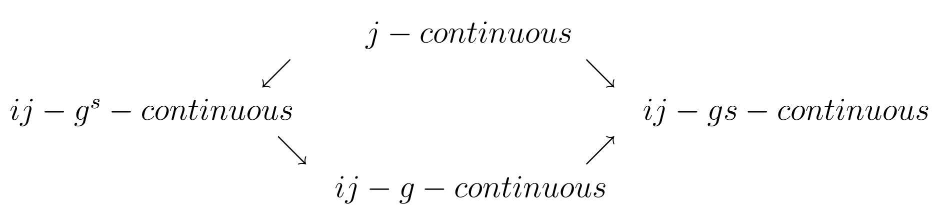

Figure (3)

Theorem 6.7 *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \nu_1, \nu_2)$ be two functions, then:*

(i) If g is j-continuous and f is ij-g^s-continuous, then $g \circ f$ is ij-g^s-continuous.

(ii) If g is ij-g^s-irresolute and f is ij-g^s-irresolute, then $g \circ f$ is ij-g^s-irresolute

(iii) If g is ij-g^s-continuous and f is ij-g^s-irresolute, then $g \circ f$ is ij-g^s-continuous.

(iv) Let (Y, σ_1, σ_2) be P-$T_{1/2}$-space, g is ij-g-continuous and f is ij-g^s-continuous. Then $g \circ f$ is ij-g^s-continuous

Proof. obvious.

Remark 6.8 *In general, the composition of two ij-g^s-continuous functions is not ij-g^s-continuous.*

Example 6.9 *Let $X = Y = Z = \{a, b, c\}$,$\tau_1 = \{\phi, \{a, b\}, X\}$,$\tau_2 = \{\phi, \{a\}, \{a, c\}, X\}$, $\sigma_1 = \sigma_2 = \{\phi, \{a, c\}, Y\}$,$\nu_1 = \{\phi, \{a\}, \{a, b\}, Z\}$ and $\nu_2 = \{\phi, \{c\}, \{a, c\}, Z\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \nu_1, \nu_2)$ be identity functions. It is easily observed that f and g are 12-g^s-continuous. But the composition function $g \circ f : (X, \tau_1, \tau_2) \to (Z, \nu_1, \nu_2)$ is not 12-g^s-continuous, since $\{a, b\}$ is ν_2-closed set in Z but not 12-g^s-closed set in X.*

Theorem 6.10 *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is bijectives, ij-g-open and ij-g^s-continuous, then f is ij-g^s-irresolute.*

Proof. *Let V be ij-g^s-closed set of Y and let $f^{-1}(V) \subset U$, where U be ij-g-open set. Clearly $V \subset f(U)$. Since $f(U)$ is ij-g-open and since V is ij-g^s-closed set in Y, then j-$cl(V) \subset f(U)$ and thus $f^{-1}(j\text{-}cl(V)) \subset U$. Since f is ij-g^s-continuous and since j-$cl(V)$ is σ_j-closed in Y, then $j\text{-}cl(f^{-1}(j\text{-}cl(V))) \subset U$ and hence $j\text{-}cl(f^{-1}(V)) \subset U$. Therefore, $f^{-1}(V)$ is ij-g^s-closed of X. Hence f is ij-g^s-irresolute.*

Theorem 6.11 *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an ij-g^s-irresolute and X is P-T_p. Then f is j-continuous.*

Proof. *Let V be a j-closed set of Y. Since V is ij-g^s-closed in Y and f is ij-g^s-irresolute, then $f^{-1}(V)$ is ij-g^s-closed in X. But X is P-T_p and so $f^{-1}(V)$ is j-closed. Hence f is j-continuous.*

Theorem 6.12 *If $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is ij-g-irresolute and j-closed, then every ij-g^s-closed set A of X, $f(A)$ is ij-g^s-closed set of Y.*

Proof. *Let A be an ij-g^s-closed set. Suppose that $f(A) \subset U$, where U is an ij-g-open in Y. Then $A \subset f^{-1}(U)$ and $f^{-1}(U)$ is ij-g-open, since f is ij-g-irresolute. Since A is ij-g^s-closed, $j\text{-}cl(A) \subset f^{-1}(U)$ and hence $f(j\text{-}cl(A)) \subset U$. Therefore, we have $j\text{-}cl(f(A)) \subset j\text{-}cl(f(j\text{-}cl(A))) = f(j\text{-}cl(A)) \subset U$, since f is j-closed. Hence $f(A)$ is ij-g^s-closed in Y.*

Theorem 6.13 *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is surjective j-closed and j-continuous, then for every ij-g^s-closed set B of $Y, f^{-1}(B)$ is ij-g^s-closed set of X.*

Proof. *Let B be an ij-g^s-closed subset of Y and $f^{-1}(B) \subset U$, where U is a j-open set of X. Since f is j-closed and by Lemma 2.6, there is a σ_j-open set V such that $B \subset V$ and $f^{-1}(V) \subset U$. Since B is ij-g^s-closed set and $B \subset V$, then j-$cl(B) \subset V$. Hence $f^{-1}(j\text{-}cl(B)) \subset f^{-1}(V) \subset U$. By Lemma 2.7, j-$cl(f^{-1}(B)) \subset U$ and hence $f^{-1}(B)$ is ij-g^s-closed set in X, since every j-open set is ij-g-open.*

Theorem 6.14 *If a space X is P-T_p and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is surjective, j-continuous and j-closed, then Y is P-T_p.*

Proof. *Let B be an ij-g^s-closed subset of Y. Then by Theorem 6.13, we have $f^{-1}(B)$ is an ij-g^s-closed subset of X. It follows by assumptions that, $f^{-1}(B)$ is j-closed and hence B is j-closed. So Y is P-T_p.*

Theorem 6.15 *If a space X is P-T_p and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is bijective, P-open, then Y is P-T_p.*

Proof. *Let $\{y\}$ be a singleton of Y. Since X is P-T_p and f is onto, for some $x \in X$ with $f(x) = y, \{x\}$ is j-open or ij-g-closed by Theorem 4.28. If the singleton $\{x\}$ is j-open, since f is P-open, then $\{y\}$ is j-open. If $\{x\}$ is ij-g-closed, then $\{y\}$ is ij-g-closed by hypothesis and Lemma 2.8. Thus Y is P-T_p, by Theorem 4.28.*

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