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Quarter Symmetric Metric Finsler Connections on Kenmotsu and P-Kenmotsu Vector Bundles

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Abstract

Quarter symmetric Finsler connections are defined on the total spaces of Kenmotsu and P-Kenmotsu vector bundles. In addition, torsion tensor properties of the connection are derived. It has been shown that the quarter symmetric Finsler connection on both Kenmotsu and P-Kenmotsu vector bundles is metrical Finsler connection.

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Introduction

Semi symmetric and quarter symmetric connections are known and well studied on a manifold [3,12]. Miron [6] introduced a sophisticated method for the study of Finsler Geometry of vector bundles. Using the techniques of Miron's theory, Atanasiu [1,2] defined the semi symmetric and quarter symmetric Finsler connections on the total space of vector bundle and studied several properties. Sai Prasad [8] defined Kenmotsu and Para Kenmotsu Finsler structures on the total space of vector bundles and called the corresponding vector bundles as Kenmotsu and P-Kenmotsu vector bundles respectively.

In this paper, Quarter symmetric Finsler connections are defined on the total spaces of Kenmotsu and P-Kenmotsu vector bundles. In addition, torsion tensor properties of the connection are derived. It has been shown that the quarter symmetric Finsler connection on both Kenmotsu and P-Kenmotsu vector bundles is metric.

1. Preliminaries

Let V(M) = {VM, π, M} be a vector bundle whose total space VM is a (n + m) dimensional c^{∞} - manifold and base space M is an n-dimensional c^{∞} - manifold.

A Finsler connection ∇ on the total space VM of a vector bundle V(M) is characterised by the horizontal part ∇^H and the vertical part ∇^V [6].

A Finsler connection ∇ on VM is said to be quarter symmetric if its torsion tensor T satisfies:

$$[T(X^H, Y^H)]^H = p^H(Y^H)\, tX^H - p^H(X^H)\, tY^H$$
$$[T(X^V, Y^V)]^V = p^V(Y^V)\, tX^V - p^V(X^V)\, tY^V \quad (1.1)$$

$\forall$ X, Y $\in T_u$ (VM) and p$\in T_u^*$ (VM).

A Finsler connection ∇ on VM is said to be metric if and only if

$$\nabla_X G^H = 0, \nabla_X G^V = 0$$

If D is the metrical Finsler connection on VM, then any quarter symmetric metric Finsler connection ∇ on VM is given by :

$$\nabla_X Y = D_X Y + p(Y)\, t X - {}'t(X, Y)P \quad (1.2)$$

$\forall$ X, Y $\in T_u$ (VM), where p is a 1-form and t is a Finsler tensor field of type $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$.

Here,

$${}'t(X,Y) = G(t(X), Y) = G(X, t(Y)) \text{ and } p(x) = G(P, X). \quad (1.3)$$

A quarter symmetric Finsler connection ∇ on VM satisfying

$$(\nabla_X p)(Y) = (\nabla_Y p)(X) \text{ i.e.,}$$
$$(\nabla^H_X p)(Y^H) = (\nabla^H_Y p)(X^H)$$
$$(\nabla^V_X p)(Y^V) = (\nabla^V_Y p)(X^V) \quad (1.4)$$

is called a special quarter symmetric Finsler connection on VM.

The Vector bundle VM with Riemannian Finsler metric G admitting a Finsler tensor field ϕ of type $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, a vector field ξ and a 1-form η satisfying:

(i) $(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0.$

(ii) $(\nabla_X \nabla_Y \eta)(Z) = [-G(X, Z) + \eta(Z)]\, \eta(Y) + [-G(X, Y) + \eta(X)\, \eta(Y)]\, \eta(Z)$

(iii) $\eta(X) = G(X, \xi)$

(iv) $\nabla_X \xi = X - \eta(X)\, \xi$; $\forall$ X, Y $\in T_u$ (VM). (1.5)

is said to possess the P-Kenmotsu Finsler structure[8]. The torsion free Finsler connection ∇satisfying (1.5) is called P-Kenmotsu Finsler connection on VM.

The Vector bundle VM with Riemannian Finsler metric G admitting a Finsler tensor field ϕ of type $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, a vector field ξ and a 1-form η satisfying:

(i) $(\nabla_X \eta)(Y) = G(X, Y) - \eta(X)\, \eta(Y)$

(ii) $\eta(X) = G(X, \xi)$ and

(iii) $(\nabla_X \eta)(Y) = \Omega(\bar{X}, Y)$; $\forall X, Y \in T_u(VM)$. (1.6)

is said to possess the SP-Kenmotsu Finsler structure[8]. The torsion free Finsler connection ∇satisfying (1.6) is called SP-Kenmotsu Finsler connection on VM.

2. Quarter symmetric Finsler connections on Kenmotsu and para Kenmotsu vector bundles

Definition(2.1): On the total space VM of a Kenmotsu vector bundle[8] we introduce a quarter symmetric Finsler connection ∇ by identifying p and t of (1.1) respectively with η and ϕ, where η is a 1 – form and ϕ is an almost contact Finsler structure on VM[11] given by the Finsler tensor field of type $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, that is, by setting

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X, Y] = \eta(Y)\,\phi X - \eta(X)\,\phi Y,$$

which give

$$[T(X^H,Y^H)]^H = \eta^H(Y^H)\,\phi X^H - \eta^H(X^H)\,\phi Y^H$$
$$[T(X^V,Y^V)]^V = \eta^V(Y^V)\,\phi X^V - \eta^V(X^V)\,\phi Y^V \qquad (2.1)$$

$\forall X, Y \in T_u(VM)$ and $\eta \in T_u^*(VM)$.

Then, for a quarter symmetric Finsler connection ∇ on VM of Kenmotsu vector bundle, we have

(a) $[T(\phi X^H, \phi Y^H)]^H = 0$; $[T(\phi X^V, \phi Y^V)]^V = 0$

(b) $[T(\phi^2 X^H, Y^H)]^H + [T(X^H, \phi^2 Y^H)]^H = -[T(X^H, Y^H)]^H$
$[T(\phi^2 X^V, Y^V)]^V + [T(X^V, \phi^2 Y^V)]^V = -[T(X^V, Y^V)]^V$

(c) $\phi^2 [T(X^H, Y^H)]^H + [T(X^H, Y^H)]^H = 0$
$\phi^2 [T(X^V, Y^V)]^V + [T(X^V, Y^V)]^V = 0$

(d) $\eta^H [T(X^H, Y^H)]^H = 0$; $\eta^V [T(X^V, Y^V)]^V = 0$ (2.2)

$\forall X, Y \in T_u(VM)$ and $\eta \in T_u^*(VM)$.

Definition (2.2): On the total space VM of a Para Kenmotsu (or) SP-Kenmotsu vector bundle[8] we introduce a quarter symmetric Finsler connection ∇ by identifying p and t of (1.1) respectively with η and ϕ, where η is a 1 – form and ϕ is an almost para contact Finsler structure on VM[9] given by the Finsler tensor field of type $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, that is, by setting

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X, Y] = \eta(Y)\,\phi X - \eta(X)\,\phi Y$$

which give

$$[T(X^H,Y^H)]^H = \eta^H(Y^H)\,\phi X^H - \eta^H(X^H)\,\phi Y^H$$
$$[T(X^V,Y^V)]^V = \eta^V(Y^V)\,\phi X^V - \eta^V(X^V)\,\phi Y^V \qquad (2.3)$$

$\forall X, Y \in T_u(VM)$ and $\eta \in T_u^*(VM)$.

It may be noted that to specify the clear difference between the symbols ϕ and η of (2.1) and (2.3), in this paper we replace ϕ, η, ξ, G of (2.3), concerning almost para contact Finsler structures, respectively by $\tilde{\phi}, \tilde{\eta}, \tilde{\xi}, \tilde{G}$.
Hence (2.3) can be written as

$$[T(X^H,Y^H)]^H = \tilde{\eta}^H (Y^H)\, \tilde{\phi} X^H - \tilde{\eta}^H (X^H)\, \tilde{\phi} Y^H$$

$$[T(X^V,Y^V)]^V = \tilde{\eta}^V (Y^V)\, \tilde{\phi} X^V - \tilde{\eta}^V (X^V)\, \tilde{\phi} Y^V \qquad (2.4)$$

$\forall$ X, Y $\in T_u$ (VM) and $\tilde{\eta} \in T_u^*$ (VM).
Then for a quarter symmetric Finsler connection ∇ on VM of P-Kenmotsu (or) SP-Kenmotsu vector bundle, we get

(a) $$\left[T\left(\tilde{\phi} X^H, \tilde{\phi} Y^H\right)\right]^H = 0; \left[T\left(\tilde{\phi} X^V, \tilde{\phi} Y^V\right)\right]^V = 0$$

(b) $$\left[T\left(\tilde{\phi}^2 X^H, Y^H\right)\right]^H + \left[T\left(X^H, \tilde{\phi}^2 Y^H\right)\right]^H = \left[T(X^H, Y^H\right]^H$$

$$\left[T\left(\tilde{\phi}^2 X^V, Y^V\right)\right]^V + \left[T\left(X^V, \tilde{\phi}^2 Y^V\right)\right]^V = \left[T(X^V, Y^V\right]^V$$

(c) $$\tilde{\phi}^2\left[T(X^H, Y^H\right]^H - \left[T(X^H, Y^H\right]^H = 0$$

$$\tilde{\phi}^2\left[T(X^V, Y^V\right]^V - \left[T(X^V, Y^V\right]^V = 0$$

(d) $$\tilde{\eta}^H \left[T(X^H, Y^H\right]^H = 0 ; \tilde{\eta}^V \left[T(X^V, Y^V\right]^V = 0 \qquad (2.5)$$

$\forall$ X, Y $\in T_u$ (VM) and $\tilde{\eta} \in T_u^*$ (VM).

Theorem (2.1) : Let ∇ be a quarter symmetric Finsler connection on the total space VM of a Kenmotsu vector bundle. Then for every X,Y$\in T_u$(VM) there exists a Finsler tensor field A such that

$$\overset{2}{\phi}\left[A(X^H, Y^H)\right]^H + \left[A(X^H, Y^H)\right]^H = 0$$

$$\overset{2}{\phi}\left[A(X^V, Y^V)\right]^V + \left[A(X^V, Y^V)\right]^V = 0$$

Proof: If we put $\left[A(X^H, Y^H)\right]^H = \phi\left[T(Y^H, X^H)\right]^H$, then from (2.2) c we have

$$\phi\left[A(X^H, Y^H)\right]^H = \phi^2\left[T(Y^H, X^H)\right]^H = -\left[T(Y^H, X^H)\right]^H$$

Therefore,

$$\phi^2\left[A(X^H, Y^H)\right]^H = -\phi\left[T(Y^H, X^H)\right]^H = -\left[A(X^H, Y^H)\right]^H$$

which implies

$$\phi^2\left[A(X^H, Y^H)\right]^H + \left[A(X^H, Y^H)\right]^H = 0$$

Similarly the other part holds.

Theorem (2.2): Let ∇ be a quarter symmetric Finsler connection on VM of a P-Kenmotsu (or) SP-Kenmotsu vector bundle. Then for every $X, Y \in T_u$ (VM), there exists a Finsler tensor field A such that

$$\tilde{\phi}^2 \left[A(X^H, Y^H)\right]^H = \left[A(X^H, Y^H)\right]^H$$
$$\tilde{\phi}^2 \left[A(X^V, Y^V)\right]^V = \left[A(X^V, Y^V)\right]^V$$

Proof: If we put $\left[A(X^H, Y^H)\right]^H = \tilde{\phi}\left[T(Y^H, X^H)\right]^H$ then from (2.5)c, we get

$$\tilde{\phi}\left[A(X^H, Y^H)\right]^H = \left[T(Y^H, X^H)\right]^H .$$

Therefore,

$$\tilde{\phi}^2 \left[A(X^H, Y^H)\right]^H = \tilde{\phi} \left[T(Y^H, X^H)\right]^H = \left[A(X^H, Y^H)\right]^H .$$

Similarly the second part.

3. Quarter symmetric metric Finsler connection on Kenmotsu vector bundle

Let D be a metrical Finsler connection and ∇ be quarter symmetric Finsler connection on VM of Kenmotsu vector bundle. Then

$$\nabla_X Y = D_X Y + H(X, Y) \qquad \forall\ X, Y \in T_u(VM) \tag{3.1}$$

where

$$H(X, Y) = \frac{1}{2}[T(X, Y) + P(X,Y) + P(Y, X)] \tag{3.2}$$

and

$$G(P(X,Y), Z) \ \underline{\underline{def}}\ G(T(Z, X),Y) \tag{3.3}$$

From (2.1) and (3.3) we get

$$P(X, Y) = -\eta(X)\,\phi\, Y - \Omega(X, Y)\xi \tag{3.4}$$

where $\eta(X) = G(X, \xi)$ and $\Omega(X, Y) = G(\phi X, Y) = -G(X, \phi Y)$ (3.5)

Now from (2.1), (3.2), (3.4) and (3.5) we get

$$H(X, Y) = -\eta(X)\,\phi Y \tag{3.6}$$

Then from (3.1) ∇ is given as:

$$\nabla_X Y = D_X Y - \eta(X)\,\phi Y \tag{3.7}$$

Theorem (3.1): The quarter symmetric Finsler connection ∇ on the total space VM of a Kenmotsu vector bundle, given by (3.7), is metric.

Proof: We have

$$\nabla_X (G(Y, Z)) = (\nabla_X G)(Y, Z) + G(\nabla_X Y, Z)+G(Y, \nabla_X Z)$$

Using (3.7) in the above equation, we get $(\nabla_X G)(Y,Z) = 0$, which shows that ∇ is metric.

Proposition (3.1): The Finsler tensor field H(X, Y) satisfies:

$$[H(\xi^H, Y^H)]^H = [T(\xi^H, Y^H)]^H$$
$$[H(\xi^V, Y^V)]^V = [T(\xi^V, Y^V)]^V$$
$$G[T(X^H, Y^H)^H, \xi^H] = 0 = G[T(X^V, Y^V)^V, \xi^V] \quad (3.8)$$

Proof: From equations (2.1), (3.5),and (3.6), we have (3.8).

Theorem (3.2): The quarter symmetric metric Finsler connection ∇ on the total space VM of Kenmotsu vector bundle with Kenmotsu Finsler connection D satisfies

$$(\nabla^H_X \eta)(Y^H) + (\nabla^H_Y \eta)(X^H) = 2G(\phi X^H, \phi Y^H)$$
$$(\nabla^V_X \eta)(Y^V) + (\nabla^V_Y \eta)(X^V) = 2G(\phi X^V, \phi Y^V) \quad (3.9)$$

$\forall$ X, Y $\in T_u(VM)$

Proof: From (3.7) we have

$$(\nabla_X \eta)(Y) = (D_X \eta)(Y)$$

Therefore,

$$(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) = (D_X \eta)(Y) + (D_Y \eta)(X) = 2G(\phi X, \phi Y);$$

which gives (3.9) since ∇ is a Finsler connection on VM.
[Since D is a Kenmotsu Finsler connection, we have

$$(D_X \eta)(Y) + (D_Y \eta)(X) = 2G(\phi X, \phi Y)]$$

4. Quarter symmetric metric Finsler connection on P-Kenmotsu vector bundle

Let D be a metrical Finsler connection and let ∇ be quarter symmetric Finsler connection on VM of P-Kenmotsu (or) SP–Kenmotsu vector bundle. Then

$$\nabla_X Y = D_X Y + H(X, Y); \qquad \forall X, Y \in T_u (VM) \quad (4.1)$$

where

$$H(X, Y) = \frac{1}{2}\{T(X, Y) + P(X,Y) + P(Y, X)\} \quad (4.2)$$

and

$$G(T(Z,X), Y) = G(P(X,Y), Z) \quad (4.3)$$

Then from (2.4) and (4.3), we get

$$P(X, Y) = \tilde{\eta}(X)\,\tilde{\phi}Y - \tilde{\Omega}(X, Y)\,\tilde{\xi}. \tag{4.4}$$

where $\tilde{\eta}(X) = \tilde{G}(X, \tilde{\xi})$ and

$$\tilde{\Omega}(X, Y) = \tilde{G}(\tilde{\phi}X, Y) = \tilde{G}(X, \tilde{\phi}Y) \tag{4.5}$$

Now from equations (2.4), (4.2), (4.4) and (4.5) we get

$$H(X, Y) = \tilde{\eta}(Y)\,\tilde{\phi}X - \tilde{\Omega}(X, Y)\,\tilde{\xi} \tag{4.6}$$

Then from (4.1) and (4.6), ∇ can be written as:

$$\nabla_X Y = D_X Y + \tilde{\eta}(Y)\,\tilde{\phi}X - \tilde{\Omega}(X, Y)\,\tilde{\xi}. \tag{4.7}$$

Theorem (4.1): The quarter symmetric Finsler connection on the total space VM of a P-Kenmotsu (or) SP-Kenmotsu vector bundle, given by (4.7), is metric.

Proof: We have

$$\nabla_X (G(Y, Z)) = (\nabla_X G)(Y, Z) + G(\nabla_X Y, Z) + G(Y, \nabla_X Z)$$

Using (4.7) in the above equation, we get $(\nabla_X G) = 0$ which shows that ∇ is metric.

Proposition(4.1): The Finsler tensor field H(X, Y), given by (4.6), satisfies:

$$[H(X^H, \tilde{\xi}^H)]^H = [T(X^H, \tilde{\xi}^H)]^H$$

$$[H(X^V, \tilde{\xi}^V)]^V = [T(X^V, \tilde{\xi}^V)]^V$$

$$G\{[T(X^H, Y^H)]^H, \tilde{\xi}^H\} = 0 = G\{[T(X^V, Y^V)]^V, \tilde{\xi}^V\} \tag{4.8}$$

Proof: From (2.4), (4.5) and (4.6) we have (4.8).

Theorem (4.2): On the total space VM of SP-Kenmotsu vector bundle with quarter symmetric metric Finsler connection ∇, we have

$$(\nabla^H_X \tilde{\eta})(Y^H) - (\nabla^H_Y \tilde{\eta})(X^H) = 0$$

$$(\nabla^V_X \tilde{\eta})(Y^V) - (\nabla^V_Y \tilde{\eta})(X^V) = 0 \tag{4.9}$$

$\forall$ X, Y $\in T_u(VM)$

Proof: From (4.7), we have

$$(\nabla_X \tilde{\eta})(Y) = (D_X \tilde{\eta})(Y) - \tilde{\Omega}(X, Y)$$

Hence for SP-Kenmotsu Finsler vector bundle, using (4.5), we get

$$(\nabla_X \tilde{\eta})(Y) = \tilde{G}(\tilde{\phi}X, \tilde{\phi}Y) - \tilde{G}(\tilde{\phi}X, Y).$$

Similarly,

$$(\nabla_Y \tilde{\eta})(X) = \tilde{G}(\tilde{\phi}Y, \tilde{\phi}X) - \tilde{G}(\tilde{\phi}Y, X).$$

On subtracting these two and using (4.5), we get (4.9).

Conclusion. It is found from the above result that ∇ is a special quarter symmetric metric Finsler connection on SP-Kenmotsu vector bundle. Similarly, we can also prove that "the quarter symmetric metric Finsler connection ∇ on the total space VM of a P-Kenmotsu vector bundle is special".

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