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Existence of Solutions for a Second Order Neumann Boundary Value Problem

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Abstract

We study the existence of solutions for the second-order Neumann boundary value problem $-u'' + u = f(x,u)$, $u'(a) = u'(b) = 0$, $x \in [a,b]$ under suitable assumptions on the function f. Our approach is based on the Green's function method and on well known fixed point theorems.

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1 Introduction

In this paper, we deal with the following equation with boundary homogeneous conditions

$$-u'' + u = f(x,u) \quad , \quad u'(a) = u'(b) = 0 \quad x \in [a,b] \tag{1}$$

This is a second-order Neumann problem, many applications of which arise in different contexts, e.g. in various typical mathematical physics approaches. By using the Green's function technique (see [2]) we'll see that this problem, as well as the analogous Dirichlet's form, is equivalent to an integral equation of Fredholm type. Therefore, such a Neumann problem (1) can be reduced to a fixed point problem for a given functional. The aim of the present paper is to establish the existence of the solution, under suitable assumptions on f, by using the Schauder-Tychonoff theorem (see for instance [4]). Applications of the Schauder-Tychonoff theorem to differential equations have been discussed by many authors (see for instance Leray [6] and Bonsall [1]). For the sake of completeness we include at first, under peculiar hypothesis on f, the proof of the existence and uniqueness of the solution of (1) by using the Banach-Caccioppoli's fixed point theorem which, as known, applies to contraction (distance-diminishing) maps of a complete metric space into itself. In such a case, we present also a construction of the solution of (1) by means of successive approximations method.

2 The second-order Neumann Problem

As it's well known by the literature, problems of the kind (1) can be solved by preliminarily considering the following Neumann problem

$$-u'' + u = v \quad , \quad u'(a) = u'(b) = 0 \quad , \quad x \in [a,b] \tag{2}$$

by assuming, of course, $v \in L^1([a,b])$.

The solution of (2) can be written in the form

$$u(x) = \int_a^b G(x,t)v(t)dt \tag{3}$$

where

$$G(x,t) = g(x,t) - \frac{e^x}{e^{2b} - e^{2a}} \left\{ e^b \left(\frac{\partial g}{\partial x}\right)_{x=b} - e^a \left(\frac{\partial g}{\partial x}\right)_{x=a} \right\} +$$

$$+ \frac{e^{-x}}{e^{-2b} - e^{-2a}} \left\{ e^{-b} \left(\frac{\partial g}{\partial x}\right)_{x=b} - e^{-a} \left(\frac{\partial g}{\partial x}\right)_{x=a} \right\}$$

The function $g(x,t)$ should be solution of the following differential equation in the distributional sense

$$-\frac{\partial^2 g}{\partial x^2}(x,t) + g(x,t) = \delta(x-t)$$

whose solution, by means of the Fourier transform technique, is $g(x,t) = \frac{1}{2}e^{-|x-t|}$. Consequently,

$$G(x,t) = \frac{\cosh(b-\xi)\cosh(a-\eta)}{\sinh(b-a)} \tag{4}$$

with $\xi = max(x,t)$, $\eta = min(x,t)$ and $(x,t) \in D = [a,b] \times [a,b]$. We remark that the $G(x,t)$ represents the Green's function for the problem (2) and satisfies the conditions

$$\left(\frac{\partial G}{\partial x}\right)_{x=a} = \left(\frac{\partial G}{\partial x}\right)_{x=b} = 0 \tag{5}$$

The function $G(x,t)$ is continuous and positive over all D. Moreover, one has both $\frac{\partial G}{\partial x}(x,t) \neq 0$ and $\frac{\partial G}{\partial t}(x,t) \neq 0$, $\forall\ (x,t) \in intD$ and also $G(x,t) \leq \coth(b-a)$, $\forall\ (x,t) \in D$.

2.1 An existence and uniqueness theorem

Let $f(x,u)$ be a measurable and bounded function defined in $[a,b] \times R$. Thus, by (3) the boundary value problem (1) is equivalent to the following integral equation of Fredholm type

$$u(x) = \int_a^b G(x,t) f(t, u(t)) dt \tag{6}$$

We suppose now that $f(x,u)$ is also a Lipschitz function with respect to u, uniformely with respect to x. Denoting by L a Lipschitz constant, we prove that, provided

$$L < \frac{\tanh(b-a)}{b-a} = \Lambda\ , \tag{7}$$

the problem (1) has a unique solution. We remark that if define a functional $\Phi : C^0([a,b]) \longrightarrow C^0([a,b])$ as $\Phi(u(x)) = \int_a^b G(x,t) f(t,u(t)) dt$ the (6) becomes a fixed point problem. Hence, the assumption (7) implies that the functional Φ is a contraction map in $C^0([a,b])$ with L/Λ a Lipschitz constant and, as it's well known, the Banach-Caccioppoli's fixed point theorem assures that Φ has a unique fixed point $u \in C^0([a,b])$ satisfying the integral equation (6).

In order to construct such a solution, let us consider the following recursive sequence

$$u_n(x) = \int_a^b G(x,t) f(t, u_{n-1}(t)) dt \quad , \quad \forall\ n\ \in\ N \tag{8}$$

where

$$u_o(x) = 0$$

It results that, in our hypothesis, the $u_n(t)$ are continuous functions for every $t \in [a,b]$ and from (5) it follows also that the boundary conditions of (1) are satisfied at each step n . Then, by setting $H = max_{t\in[a,b]}|f(t, 0)|$, the (8) implies

$$|u_k(x) - u_{k-1}(x)| \leq \frac{H}{L}[L/\Lambda]^k \quad , \quad \forall\ k\ \in\ N \quad , \forall\ x\ \in\ [a,b]$$

This implies the total and hence the uniform convergence of the $\{u_n(x)\}_n$ sequence towards a continuous function $u(t)$. Consequently, the sequence $\{z_n(x,t) = G(x,t)f(t, u_{n-1}(t))\}_n$ converges uniformely , $\forall\ (x, t) \in D$, towards the limit function $z(x, t) = G(x, t)f(t, u(t))$ being

$$|z_n(x,t) - z(x,t)| \leq \coth(b-a)L|u_n(t) - u(t)| \quad ,\forall\ n\ \in\ N\ ,\ \ \forall\ (x,t)\ \in\ D$$

Therefore, by the *Uniform Convergence Theorem*, it follows from (8) that the limit function $u(t)$ is just the solution of (6). Such a solution is obviously unique. Indeed, arguing by contradiction, let $w(t)$ be another solution of the problem (1) . From (6) clearly follows the inequality

$$|u(x) - w(x)| \leq \frac{L}{\Lambda} max_{x\in[a,b]}|u(x) - w(x)| \quad , \quad \forall\ x\ \in\ [a,b]$$

which by (7) obviously leads to an absurdity.

2.2 An existence theorem

Let us assume, now that

i) the function $x \longmapsto f(x, u)$ is measurable $\forall\ u \in R$

ii) the function $u \longmapsto f(x, u)$ is continuous for almost every $x \in [a, b]$

iii)there exist two functions $p(x)$ and $s(x) \in L^1([a, b])$ such that, for almost every $x \in [a, b]$, for every $u \in R$ and for every $r > 0$ one has

$$|f(x,u)| \leq p(x)|u|^r + s(x)$$

with the constraint

$$\| p \|_{L^1} + \| s \|_{L^1} < \tanh(b-a)$$

and prove the existence of solutions of (1).

By defining the functional

$$\Psi(v(x)) \equiv \Psi(v)(x) = f\left(x, \int_a^b G(x,t)v(t)dt\right)$$

the problem (1), taking into account also the (2) and (3), becomes the fixed point problem $v(x) = \psi(v)(x)$. Let us choose γ such that

$$\frac{\| s \|_{L^1}}{\tanh(b-a) - \| p \|_{L^1}} < \gamma \tag{9}$$

where γ is greater than 1 for $0 < r < 1$ and smaller than 1 for $r \geq 1$. We observe that if $|u| < \gamma$ we have $|f(x,u)| \leq p(x)\gamma + s(x)$, $\forall\ x \in [a,b]$. Then, by defining $M(x) = sup_{|u|<\gamma}|f(x,u)|$, one clearly has

$$\| M \|_{L^1} \leq \| p \|_{L^1}\gamma + \| s \|_{L^1} < \gamma \tanh(b-a)$$

Let us consider now the set

$$K = \left\{v(x) \in L^1([a,b]) \colon |v(x)| \leq M(x)\ \ a.e.\ \ in\ [a,b]\right\}$$

K is, of course, nonempty, convex and by the Dunford-Pettis theorem (see for instance [3] Theorem 1, page 101), it is also weakly compact. Besides if $v \in K$, one has $\| v \|_{L^1} < \gamma \tanh(b-a)$. Let us prove that $\Psi(K) \subseteq K$. Since,

$$\left|\int_a^b G(x,t)v(t)dt\right| \leq \int_a^b |G(x,t)||v(t)|dt \leq \coth(b-a)\| v \|_{L^1} < \gamma$$

we have

$$|\Psi(v)(x)| = \left|f\left(x, \int_a^b G(x,t)v(t)dt\right)\right| \leq sup_{|u|<\gamma}|f(x,u)| = M(x)$$

i.e. $\Psi(v)(x) \in K$.

Let us prove now that the operator Ψ is weakly continuous. Owing to the weak compactness of K, we need only to verify that $gr(\Psi)$ is weakly closed in $K \times K$. According to [5], Theorem 7 page 313, it suffices to show that $gr(\Psi)$ is sequentially weakly closed.

Let v and $\{v_n(x)\}_n$ be a function and a sequence in K such that $v_n(x) \rightharpoonup v(x)$ in $L^1([a,b])$ respectively. We note also that $\Psi(v_n)(x) = f(x, \int_a^b G(x,t)v_n(t)dt)$ converges almost everywhere in $[a,b]$ to $f(x, \lim_{n\to+\infty}\int_a^b G(x,t)v_n(t)dt)$ which is equal to $f(x, \int_a^b G(x,t)v(t)dt) \equiv \Psi(v)(x)$ being $\varphi(\omega) = \int_a^b G(x,t)\omega(t)dt$ a linear and continuous functional in K for a fixed x.

Moreover $\Psi(v_n)(x)$ is measurable $\forall\, n \in N$ and $|\,\Psi(v_n)(x)| \leq M(x)\,\forall\, n \in N$ almost everywhere in $[a,b]$. From the Lebesgue's dominated convergence theorem, we obtain that $\lim_{n\to+\infty} \Psi(v_n)(x) = \Psi(v)(x)$ in $L^1([a,b])$ and consequently $\Psi(v_n)(x) \rightharpoonup \Psi(v)(x)$. At this point, we are allowed to apply the Schauder-Tychonoff fixed point theorem to Ψ. Therefore, it does exist $u \in K$ such that $u = \Psi(u)$. This concludes the proof of the theorem.

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