Preserving SPR0 Functions and Stability via SSPM Maps, Multiplier Sequences and Operators of Polynomials

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Abstract

In this paper we present new methods for generating Strong Stability Preserving Maps (SSPM’s). A first method is based on substitutions of strictly positive real functions with zero relative degree (SPR0 functions) in a Hurwitz polynomial, which consequently generates the family of the concerned operators. For our second method, based on sufficient conditions for the closeness of SPR0 functions by the Hadamard product of a class of polynomials with the numerator and denominator of the SPR0 function, can be interpreted as a vector-matrix product. With the introduction of a non-negative multiplier sequence, we show how the product between the non-negative multiplier sequence and a Hurwitz polynomial can be represented as a SSPM. It is also presented a relationship between stable polynomials and stable polynomials of the Jacobi class by the use of a multiplier sequence of the decreasing kind. In the last section, using generalized products of polynomials, stability and SPR0 functions are preserved by operators that preserve simple-rootedness and negative real roots. An extension to families of polynomials depending on parameters is given, and also an extension to operator of deepness-\(k\) on stable polynomials.

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1 Introduction

The notion of a matrix as a stability preserving map (SPM) was introduced in [7]; it is basically a class of linear transformations whose mapping preserves the stability of the system. If there exists some stable $n$-vector of fixed order that it is mapped to a stable $m$-vector of fixed order, then we have a SPM. If the linear transformation maps every stable $n$-vector to a stable $m$-vector, then we have a Strong SPM (SSPM). A SPM, represented by a matrix, maps polynomial coefficients to polynomial coefficients. This concept can be used to characterize a fixed order controller to simultaneously stabilize a finite number of plants. In addition, in [7], it is developed a number of tests to check whether a given matrix is a stability preserving map or not. These tests, from the perspective shown in [7], can be used as a foundation for the methods of synthesis/design of robust controllers.

In this paper the results in [7, 8] are related by the substitution of a strictly positive real function with zero relative degree on a Hurwitz stable polynomial, thus proceeding to obtain a linear operator SSPM, and as a consequence a family of SSPM's operators is obtained. In [7] it is presented exclusively the cases for SPM's.

In addition, this paper presents sufficient conditions concerning the close-ness of SPR0 functions when applying the Hadamard product to the numerator and the denominator polynomials, which it is interpreted in terms of a vector-matrix product by the use of concepts presented by Talbot in [15].

A different methodology for generating stability preserving linear operators SSPM is presented by representing them as a vector-matrix product, the multiplication term by term of a non-negative multiplier sequence with a Hurwitz stable polynomial [3]. This can be done in both numerator and denominator of a SPR0 function which in turn produces another rational function which is SPR0.

Also, illustrated by two examples, it is presented a relationship between stable polynomials and stable polynomials of the Jacobi class using multiplier sequences of the decreasing kind that can also be represented as a SSPM operators.

Finally, based on the results from [2], we introduce a new class of products and operators over real or complex polynomials as well as rational functions. This products and operators preserve stable polynomials and SPR0 functions. Moreover, using the result from [14], we extend the preservation of stability to the case when parametric dependencies exists. A new class of deepness-$k$ operators are presented, which also preserve stability. An example based on the derivative operator and the Gauss-Lucas theorem allows to show the use of the different results presented in Section 5.
2 Preliminaries

This section presents definitions and results which will be used throughout the paper.

2.1 Notation:

\( \mathbb{Z} \): integer numbers.
\( \mathbb{Z}^+ \): positive integer numbers.
\( \mathbb{R} \): field of real numbers.
\( \mathbb{C} \): field of complex numbers.
\( \mathbb{C}^+ \): right-half complex plane.
\( \mathbb{C}^- \): left-half complex plane.
\( j \): imaginary unit, \( \sqrt{-1} \).
\( \text{Re}[z] \): real part with \( z \in \mathbb{C} \), \( z = \sigma + j\omega \), \( (\sigma, \omega \in \mathbb{R}) \); \( \text{Re}[z] = \sigma \).
\( s \): Laplace complex variable.
\( \mathbb{R}[s] \): the ring of real polynomials.
\( \mathbb{H} \) is the set of Hurwitz stable polynomials.
\( \deg[p(s)] = n \) degree of the polynomial \( p(s) \), with \( n \in \mathbb{Z}^+ \).
\( \mathbb{R}^n \) is the vector space on \( \mathbb{R} \) with vectors of \( n \) components. Similarly, for the vector space \( \mathbb{R}^m \).

Consider a rational function

\[
G(s) = \frac{p(s)}{q(s)} = \frac{a_ns^n + \ldots + a_0}{b_ms^m + \ldots + b_0}
\]

the relative degree of \( G(s) \) is the integer

\[
\deg[p(s)] - \deg[q(s)] = m - n,
\]

when \( m - n \geq 0 \), it is said that \( G(s) \) is a proper rational function.

\( Z_c(p(s)) \) denotes the number of complex zeros of the polynomial \( p(s) \) counting multiplicities.

2.2 Preliminary definitions

In this subsection, we give a list of basic definitions for this work.

**Definition 2.1 ([7]).** The function \( f_A \) is called a matrix strong stability preserving map (SSPM) if every stable \( n \)-vector \( \phi \) is mapped to a stable \( m \)-vector \( \psi = f_A(\phi) \equiv \phi A \), where \( A \in \mathbb{R}^{n \times m} \) is a matrix of \( n \times m \) with entries in \( \mathbb{R} \) and \( \phi \) is the \( n \)-vector corresponding to the coefficients of some Hurwitz stable polynomial.
Definition 2.2. An $n$th degree polynomial $p(s) = s^n + a_1s^{n-1} + \cdots + a_n$ with real coefficients is Hurwitz stable if all its roots have negative real part.

Definition 2.3 ([12]). The Hadamard product of two polynomials, of the form $p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0$ and $q(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0$, in $\mathbb{R}[s]$, is defined to be the polynomial $p \circ q = a_kb_k s^k + a_{k-1} b_{k-1} s^{k-1} + \cdots + a_0 b_0$ where $k = \min(n, m)$.

Definition 2.4 ([11]). The even and the odd parts of a polynomial evaluated in $j \omega$, $p^e(\omega)$ and $p^o(\omega)$, are both polynomials in $\omega^2$, and their sets of roots will always be symmetric with respect to the origin of the complex plane, by defining: $p^e(\omega) \triangleq p^{\text{even}}(j \omega) = p_0 - p_2 \omega^2 + p_4 \omega^4 - \cdots$ and $p^o(\omega) \triangleq p^{\text{odd}}(j \omega) = p_1 - p_3 \omega^2 + p_5 \omega^4 - \cdots$.

Let $p_0(s)$ and $\delta(s)$ be polynomials and define $S$ to be the set of $\lambda \in \mathbb{R}$ for which $p_0(s) + \lambda \delta(s)$ is stable. For certain polynomials $\delta(s)$, known as convex directions, the set $S$ is an interval.

Definition 2.5. A rational function $G(s)$ is said to be positive real function if $G(s) \in \mathbb{R}$ when $s \in \mathbb{R}$, analytic in the right-half complex plane and the real part of the function evaluated on the imaginary axis is strictly positive, i.e.

$$\text{Re} \{ G(s) |_{s=j\omega} \} > 0, \forall \omega \in \mathbb{R}.$$

Now we recall some results which will be useful in the sequel.

2.3 Useful results

Lemma 2.6 ([11]). Let $G(s) = \frac{b(s)}{a(s)}$ be a rational function. Then $G(s)$ is strictly positive real of zero relative degree (SPR0) if and only if the following three conditions are satisfied:

1. $\text{Re}[G(0)] > 0$,
2. $p(s)$ is Hurwitz stable,
3. $q(s) + j \alpha p(s)$ is Hurwitz stable for all $\alpha \in \mathbb{R}$.

Notice that a SPR0 function is equivalent to a strictly passive function.

The Lemma 2.7 is proved for completeness, and it is slightly different from a result in [15].

Lemma 2.7 ([15]). If $A$ is a linear operator which maps Hurwitz stable polynomials in Hurwitz stable polynomials and $\frac{\Phi}{\tilde{\psi}}$ is an SPR0 function, then $\frac{\Phi^A}{\tilde{\psi}^A}$ is a SPR0 function.
Proof. The proof is based on Lemma 2.6. First, we establish the explicit relation between the rational function
\[
G(s) = \frac{p(s)}{q(s)} = \frac{a_n s^n + \cdots + a_0}{b_n s^n + \cdots + b_0}
\]
and the vector representation \( \phi \). This relation is given by
\[
\phi \equiv (a_n, \ldots, a_0) \\
\psi \equiv (b_n, \ldots, b_0).
\]
The notation \( \phi/\psi \) represent the rational function \( \frac{p(s)}{q(s)} \) only using their coefficients, and the notation \( \phi_A/\psi_A \) represent the rational function
\[
\frac{w(s)}{z(s)} = \frac{w_n s^n + \cdots + w_0}{z_n s^n + \cdots + z_0}
\]
where
\[
\phi A = (w_n, \ldots, w_0) = (a_n, \ldots, a_0) A \\
\psi A = (z_n, \ldots, z_0) = (b_n, \ldots, b_0) A
\]
and \( w_i \) is an scalar product of the vector \( \phi \) and the \( i \)-column of the matrix \( A_{(n+1) \times m} \) and similarly for \( z_i \) with the vector \( \psi \) and the \( i \)-column of the matrix \( A_{(n+1) \times m} \) both for \( 1 \leq i \leq n + 1 \).

We need to show that the conditions for Lemma 2.6 are fulfilled. Items 1 and 2 from Lemma 2.6 are trivial.

To show Item 3 from Lemma 2.6 we proceed as follows: By hypothesis the following polynomial \( q(s) + j \alpha p(s) \) is Hurwitz stable for all \( \alpha \in \mathbb{R} \). Now as \( A \) is a linear operator which maps Hurwitz stable polynomials in Hurwitz stable polynomials, then the vector representation based in coefficients of the polynomial \( q(s) + j \alpha p(s) \) is \( \theta = \phi + j \alpha \psi \) and applying the operator \( A \) to \( \theta \), we obtain that \( \theta A = \phi A + j \alpha \psi A \) is Hurwitz stable for all \( \alpha \in \mathbb{R} \). Therefore, Item 3 from Lemma 2.6 is satisfied. \( \square \)

Lemma 2.8 ([11]). Given an univariate polynomial \( f(s) \) with complex coefficients with decomposition:
\[
f(js) = (b_0 s^n + \cdots + b_n) + j (a_0 s^n + \cdots + a_n);
\]
where both \( b_0 s^n + \cdots + b_n \) and \( a_0 s^n + \cdots + a_n \) are real polynomials, then \( f(s) \) is stable if and only if all even-order minors of
\[
\Delta_{2p} = \begin{bmatrix}
a_0 & a_1 & \cdots & a_{2p-1} \\
b_0 & b_1 & \cdots & b_{2p-1} \\
0 & a_0 & \cdots & a_{2p-2} \\
0 & b_0 & \cdots & b_{2p-2} \\
\vdots & \vdots & \ddots & \vdots 
\end{bmatrix}
\]
where \( p = 1, 2, \ldots, n \), \( a_k = b_k = 0 \) for \( k > n \) are positive.
Lemma 2.9 ([8]). Let $Q(s)$ be a SPR0 function with arbitrary coefficients, i.e.:

$$Q(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0}$$

and let

$$\phi(s) = \phi_m s^m + \phi_{m-1} s^{m-1} + \cdots + \phi_0$$

be an Hurwitz stable polynomial. Then,

$$\psi(s) = (b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0)^m \phi(Q(s)),$$

is a Hurwitz stable polynomial.

3 SSM maps and passivity

In this section we present the main results of this paper. We begin showing that it is possible to generate SSM maps.

Let $Q(s)$ be SPR0 function with arbitrary coefficients, i.e.:

$$Q(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0}$$

and let $\phi(s)$ be a Hurwitz stable polynomial, i.e.

$$\phi(s) = \phi_m s^m + \phi_{m-1} s^{m-1} + \cdots + \phi_0$$

Then, the Hurwitz stable polynomial

$$\psi(s) = (b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0)^m \phi(Q(s)),$$

can be represented as a SSM, by using the following procedure:

1. Perform the substitution:

$$\psi(s) = (b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0)^m \phi(Q(s))$$

$$= (b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0)^m \phi \left( \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0} \right)$$

$$= (b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0)^m \sum_{i=0}^{m} \phi_i \left( \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0} \right)^i$$

$$= \sum_{i=0}^{m} \phi_i \left( a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 \right)^i \left( b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0 \right)^{m-i}$$
This procedure will be shown with an example.

**Example 3.1.** Let \( Q(s) = \frac{as^2 + bs + c}{ds^2 + es + f} \) be SPR0 rational function and \( \phi(s) = \phi_3 s^3 + \phi_2 s^2 + \phi_1 s + \phi_0 \in \mathbb{H} \). Then:

\[
\psi(s) = \left( ds^2 + es + f \right)^3 \phi(Q(s)) = \left( ds^2 + es + f \right)^3 \phi\left( \frac{as^2 + bs + c}{ds^2 + es + f} \right) = \sum_{i=0}^{3} \phi_i \left( \frac{as^2 + bs + c}{ds^2 + es + f} \right)^{3-i}
\]

The resulting polynomial in \( s \) has coefficients that are linear combinations of the coefficients of the polynomial \( \phi(s) \), then we can write \( \psi = \phi A \) with \( \phi = [\phi_3 \cdots \phi_0] \) and \( A \in \mathbb{R}^{4 \times 7} \) given by

\[
\psi(s) = \begin{bmatrix} \phi_3 \\ \phi_2 \\ \phi_1 \\ \phi_0 \end{bmatrix}^T \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{21} & m_{22} & m_{23} & m_{24} & m_{31} & m_{32} & m_{33} & m_{34} & m_{41} & m_{42} & m_{43} & m_{44} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{61} & m_{62} & m_{63} & m_{64} & m_{71} & m_{72} & m_{73} & m_{74} \end{bmatrix}^T = \begin{bmatrix} \psi_6 \\ \psi_5 \\ \psi_4 \end{bmatrix}
\]

with \( m_{11} \triangleq a^3, m_{12} \triangleq a^2, m_{13} \triangleq ad^2, m_{14} \triangleq d^3, m_{21} \triangleq 3a^2b, m_{22} \triangleq a^2e+2abd, m_{23} \triangleq 2acd+bd^2, m_{24} \triangleq 3ed^2, m_{31} \triangleq 3a^2c+3ab^2, m_{32} \triangleq a^2f+2abe+2acd+b^2d, m_{33} \triangleq 2adf+ae^2+2bed, m_{34} \triangleq 3d^2f+3e^2d, m_{41} \triangleq 6abc+b^3, m_{42} \triangleq 2abf+2ace+b^2e+2b, m_{43} \triangleq 2ae+f+2df+be^2+2ced, m_{44} \triangleq 6edf+e^3, m_{51} \triangleq 3ac^2+3b^2c, m_{52} \triangleq 2acf+b^2f+2bec+c^2d, m_{53} \triangleq af^2+2bef+2cde+ce^2, m_{54} \triangleq 3df^2+3e^2f, m_{61} \triangleq 3bc^2, m_{62} \triangleq 2bcf+c^2e, m_{63} \triangleq bf^2+2cef, m_{64} \triangleq 3ef^2, m_{71} \triangleq c^3, m_{72} \triangleq c^2f, m_{73} \triangleq cf^2, m_{74} \triangleq f^3. \)

This is a SSM that maps preserving the stability of every stable polynomial of fixed order. \( \square \)

Now we present a generalization of the Theorem 2.1 in [9].

**Proposition 3.2.** Let the rational functions \( \frac{p(s)}{q(s)} \) and \( \frac{p'(s)}{q'(s)} \) be SPR0 with the polynomials

\[
p(s) = p_0 s^n + \ldots + p_n \\
q(s) = q_0 s^n + \ldots + q_n \\
p'(s) = p'_0 s^n + \ldots + p'_n \\
q'(s) = q'_0 s^n + \ldots + q'_n
\]
and all the even-order minors of $\triangle_{2p}^\circ$ are positive, then

\[
\frac{p(s) \circ p'(s)}{q(s) \circ q'(s)} \in \text{SPR0}
\]

Where: 
\[
F(s, \alpha) = q(s) \circ q'(s) + j\alpha(p(s) \circ p'(s))
\]

\[
\triangle_{2p}^\circ = \begin{bmatrix}
\alpha^\circ_0 & \alpha^\circ_1 & \ldots & \alpha^\circ_{2p-1} \\
\beta^\circ_0 & \beta^\circ_1 & \ldots & \beta^\circ_{2p-1} \\
0 & \alpha^\circ_0 & \ldots & \alpha^\circ_{2p-2} \\
0 & \beta^\circ_0 & \ldots & \beta^\circ_{2p-2} \\
\ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]

with \( p = 1, 2, \ldots, n, \ a_k = b_k = 0 \) for \( k > n \) and

\[
\begin{align*}
\alpha^\circ_0 &= \alpha (-1)^m p_0 p'_0 \\
\beta^\circ_0 &= (-1)^m q_0 q'_0 \\
\alpha^\circ_1 &= (-1)^{m-1} p_1 p'_1 \\
\beta^\circ_1 &= (-1)^{m-1} q_1 q'_1 \\
\alpha^\circ_{n-2} &= -q_{2m-3} q'_{2m-3} \\
\beta^\circ_{n-2} &= -q_{2m-3} q'_{2m-3} \\
\alpha^\circ_{n-1} &= q_{2m-1} q'_{2m-1} \\
\beta^\circ_{n-1} &= q_{2m-1} q'_{2m-1}
\end{align*}
\]

\( \forall \alpha \in \mathbb{R} \).

**Proof.** Lemma 2.6 is used. The Item 1 and Item 2 are a consequence of the following argument. By hypothesis the functions \( \frac{p(s)}{q(s)} \), \( \frac{p'(s)}{q'(s)} \) are SPR0 functions, then \( p(s), q(s), p'(s) \) and \( q'(s) \) are Hurwitz stable. Due to the fact that the Hadamard product of Hurwitz stable polynomials is a Hurwitz stable polynomial [12], the conclusion follows.

After some algebra based on Definition 2.4 and Lemma 2.7, Item 3 is a consequence of the positivity of the minors of order even in \( \triangle_{2p}^\circ \), where \( p = 1, 2, \ldots, n, \ a_k = b_k = 0 \), for \( k > n \), therefore the polynomial \( q(s) \circ q'(s) + j\alpha(p(s) \circ p'(s)) \) is Hurwitz stable for all \( \alpha \in \mathbb{R} \).

**Remark 3.3.** Under the conditions of the above result, and as we have seen throughout the paper, \( p \circ p' = [p]A_{p'} \) where

\[
A_{p'} = \begin{bmatrix}
p'_1 & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & p'_n
\end{bmatrix}
\]

one can represent the Hadamard product performed in numerator and denominator as a vector-matrix product so,
If \( \frac{p}{q} \in \text{SPR}0 \) and \( A_{p'} \leftrightarrow p', A_{q'} \leftrightarrow q' \) then, \( \frac{pA_{p'}q}{qA_{q'}} \in \text{SPR}0 \).

Where \( A_{p'} \leftrightarrow p' \) means that \( A_{p'} \) is the diagonal matrix associated to the \( p' \) polynomial, such that we may represent it as a SSPM.

It is important to point out that this result is not a consequence of Talbot’s theorem, [15], because the linear operators in the numerator and in the denominator are different operators. Also, we point out that this is not the outcome form [12], because their result does not involve rational functions; as far as we know this is a new result for these kinds of operators.

**Corollary 3.4.** If \( \frac{p}{q} \in \text{SPR}0 \) and all the even order minors of \( \Delta_{2p}^o \) are positive, where

\[
\Delta_{2p}^o = \begin{bmatrix}
a_0^\sim & a_1^\sim & \ldots & a_{2p-1}^\sim \\
b_0^\sim & b_1^\sim & \ldots & b_{2p-1}^\sim \\
0 & a_0^\sim & \ldots & a_{2p-2}^\sim \\
0 & b_0^\sim & \ldots & b_{2p-2}^\sim \\
\ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]

for \( p = 1, 2, \ldots, n \), \( a_k = b_k = 0 \) for \( k > n \) and \( \forall \alpha \in \mathbb{R} \), with

\[
b_0^\sim = (-1)^m q_0^n, \quad a_0^\sim = \alpha (-1)^m p_0^n, \quad b_1^\sim = (-1)^{m-1} q_1^n, \quad a_1^\sim = \alpha (-1)^{m-1} p_1^n, \\
b_{n-3}^\sim = \alpha p_{2m-3}^n, \quad a_{n-3}^\sim = -q_{2m-3}^n, \\
b_{n-2}^\sim = -q_{2m-2}^n, \quad a_{n-2}^\sim = -\alpha p_{2m-2}^n, \\
b_{n-1}^\sim = -\alpha p_{2m-1}^n, \quad a_{n-1}^\sim = q_{2m-1}^n, \\
b_n^\sim = q_{2m}^n, \quad a_n^\sim = \alpha p_{2m}^n.
\]

Then, \( \frac{p[n]}{q[m]} \in \text{SPR}0 \forall n \in \mathbb{Z}^+ \), where \( p[n] = p \circ p^{[n-1]} \).

**Corollary 3.5.** Let \( A \) and \( A' \) be two SSPM linear operators and let \( G(s) \) be any SPR0 function

\[
G(s) \triangleq \frac{a(s)}{b(s)} = \frac{a_ms^m + \ldots + a_0}{b_ms^m + \ldots + b_0}
\]

If

\[
\phi A \triangleq [a_m \ldots a_0] A = [\alpha_l \ldots \alpha_0],
\]

\[
\varphi A' \triangleq [b_m \ldots b_0] A' = [\beta_l \ldots \beta_0]
\]

and all the even order minor of the matrix

\[
\Delta_{2p} = \begin{bmatrix}
\mu \alpha_0 & \mu \alpha_1 & \ldots & \mu \alpha_{2p-1} \\
\beta_0 & \beta_1 & \ldots & \beta_{2p-1} \\
0 & \mu \alpha_0 & \ldots & \mu \alpha_{2p-2} \\
0 & \beta_0 & \ldots & \beta_{2p-2} \\
\ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]

is positive, then \( \frac{\phi A}{\varphi A'} \in \text{SPR}0 \).
for \( p = 1, 2, \ldots, l, \, a_k = b_k = 0 \) and \( k > l \) and for all \( \mu \in \mathbb{R} \), be positive.

Then

\[
G'(s) \triangleq \frac{\alpha(s)}{\beta(s)} = \frac{\alpha_1 s^l + \ldots + \alpha_0}{\beta_1 s^l + \ldots + \beta_0}
\]

is a real strictly positive function with zero relative degree.

Proof. The proof is based on Lemma 2.6. Item 1 and Item 2 are satisfied. To show that Item 3 is met, let us consider the following complex polynomial

\[
(\beta_1 s^l + \ldots + \beta_0) + j\mu(\alpha_1 s^l + \ldots + \alpha_0),
\]

due to the fact that all even order minors of the matrix \( \triangle_{2p} \) are positive for \( p = 1, 2, \ldots, l, \, a_k = b_k = 0 \) and \( k > l \), for all \( \mu \in \mathbb{R} \). Then by Theorem 13 in [11] the third condition of Lemma 2.6 is satisfied. Therefore, \( G'(s) \) is a SPR0 function for any SPR0 function \( G(s) \). \( \square \)

4 Multiplier sequences and SSPM maps in passivity

Definition 4.1 ([4]). A sequence \( L = \{\gamma_k\}_{k=0}^{\infty} \) of real numbers is called a multiplier sequence if, whenever the real polynomial \( p(s) = \sum_{k=0}^{n} a_k s^k \) has only real zeros, then also the polynomial \( L[p(s)] \triangleq \sum_{k=0}^{n} \gamma_k a_k s^k \) has only real zeros.

Definition 4.2 ([4]). We say that the sequence \( L = \{\gamma_k\}_{k=0}^{\infty} \) is a complex zero decreasing sequence (CZDS), if

\[
Z_c \left( \sum_{k=0}^{n} \gamma_k a_k s^k \right) \leq Z_c \left( \sum_{k=0}^{n} a_k s^k \right)
\]

for any real polynomial \( \sum_{k=0}^{n} a_k s^k \), where \( Z_c(p) \) denotes the number of non-real zeros of the polynomial \( p \) counting multiplicities.

Lemma 4.3 ([3]). Let \( L = \{\gamma_k\}_{k=0}^{\infty} \) be a non-negative multiplier sequence (a multiplier sequence with nonnegative elements). Then \( L[p(s)] \) is a stable polynomial, whenever \( p(s) \) is a stable polynomial.

Lemma 4.4 ([4]). Let \( h(s) \) be a real polynomial. The sequence

\[
T = \{h(k)\}_{k=0}^{\infty}
\]

is a complex zero decreasing sequence (CZDS) if \( h(0) \neq 0 \) and all the zeros of \( h \) are real and strictly negative.
Any CZDS is a non-negative multiplier sequence, the converse is not true. It is possible to represent \( L[p(s)] \) in terms of an obvious vector-matrix product, i.e.,

\[
L[p(s)] = a_0 \gamma_0 + a_1 \gamma_1 + \cdots + a_k \gamma_k = \begin{bmatrix} a_0 & \cdots & a_k \end{bmatrix} \text{diag} \left( \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \end{bmatrix} \right)
\]

with \( \text{diag} \left( \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \end{bmatrix} \right) \) a diagonal matrix with entries given by \( \gamma_i, i = 0, \ldots, k \). By the use of Lemma 4.3, if \( p(s) \in \mathbb{H} \) and \( L = \{ \gamma_k \}_{k=0}^{\infty} \) is a non-negative multiplier sequence, then \( L[p(s)] \in \mathbb{H} \) and one can easily see that \( L[p(s)] \) is a SSPM.

**Corollary 4.5.** Let \( p_f(s; x) \) be a family of polynomials with negative real roots

\[
p_f(s; x) = (s + f_1(x)) \cdots (s + f_n(x))
\]

with \( f_i : \mathbb{R}^+ - \{0\} \) for \( i = 1, \ldots, n \). Let

\[
Q(s) = \frac{a(s)}{b(s)} = \frac{a_1 s^n + \cdots + a_0}{b_n s^n + \cdots + b_0},
\]

be a SPR0 rational function. Then \( L_f[Q(s)] \) is a SPR0 for all \( x \), with

\[
L_f[Q(s)] \triangleq \frac{L_f[a(s)]}{L_f[b(s)]} \triangleq \frac{p_f(n; x) a_1 s^n + \cdots + p_f(0; x) a_0}{p_f(n; x) b_n s^n + \cdots + p_f(0; x) b_0},
\]

where \( L_f = \{ p_f(k; x) \}_{k=0}^{\infty} \).

**Proof.** Corollary 4.5 is a consequence of [3, 1, 4]. By construction the family of polynomials \( p_f(s; x) \), characterizes a family of CZDS sequences, therefore it is also a family of multiplicative sequences. Then when each member of the family is applied to a stable polynomial (for the numerator and denominator polynomial of the SPR0 function) the result is a stable polynomial. As it was seen in Lemma 2.7, the family of SPR0 functions is closed under the operators represented by the multiplicative sequenced due to the fact that they are linear operators that preserve stability.

The previous result parameterizes a family of SPR0 function in terms of a generator (the family of operators); the parameters that characterize this family are those used to define the family of multiplicative sequences.
4.1 Examples

Using a multiplier sequence and a Hurwitz polynomial we will exemplify the use of Corollary 4.5 and it will be shown the same procedure for the numerator and denominator of an SPR0 function which in return produces a different SPR0 function.

Example 4.6. By Proposition 3.5 from [5] the following polynomial

\[ h(x) = (x + c)((x + a)^2 + b^2) \]

with \( a = 0.1, b = 0.2, c = 0.3 \); form a multiplier sequence \( \{h(k)\}_{k=0}^{\infty} \), where

\[ h(k) = k^3 + 0.5k^2 + 0.11k + 0.15. \]

Using the following Hurwitz polynomial

\[ p(s) = s^3 + 8s^2 + 5s + 1 \]

we obtain \( L[p(s)] \) and represent it as a SSPM. The Hurwitz polynomial is of 3\(^{rd} \) degree, thus the sequence is

\[ \{h(k)\}_{k=0}^{3} = \{0.015, 1.625, 10.235, 31.845\}. \]

We now obtain the SSPM representation

\[
\begin{bmatrix}
1 & 5 & 8 & 1 \\
\end{bmatrix}
\begin{bmatrix}
.015 & 0 & 0 & 0 \\
0 & 1.625 & 0 & 0 \\
0 & 0 & 10.235 & 0 \\
0 & 0 & 0 & 31.845 \\
\end{bmatrix}
= 
\begin{bmatrix}
.015 & 8.125 & 81.88 & 31.845 \\
\end{bmatrix}
\]

where \( p(s) = 0.015 + 8.125s + 81.88s^2 + 31.845s^3 \) which is Hurwitz stable. \( \square \)

Example 4.7. Using a SPR0 function

\[ G(s) = \frac{s^2 + 4s + 1}{s^2 + s + 2} \]

with numerator and denominator of 2\(^{nd} \) degree. Now, we generate the appropriate non-negative multiplier sequence which is

\[ \{h(k)\}_{k=0}^{2} = \{0.015, 1.625, 10.235\}. \]
We can generate a new rational function as

\[
Z'(s) = \frac{L[p(s)]_{\text{num}}}{L[p(s)]_{\text{den}}} = \begin{bmatrix}
1 & 4 & 1 \\
0 & 0 & 0 \\
0 & 1.625 & 0 \\
0 & 0 & 10.235
\end{bmatrix}
\begin{bmatrix}
0.015 & 0 & 0 \\
0 & 1.625 & 0 \\
0 & 0 & 10.235
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0.015 & 6.5 & 10.235 \\
0.03 & 1.625 & 10.235
\end{bmatrix}
\]

\[
Z'(s) = \frac{10.235s^2 + 6.5s + 0.015}{10.235s^2 + 1.625s + 0.03}
\]

the resulting rational function is a SPR0, by Lemma 2.7 \(\square\)

**Definition 4.8 ([6]).** The stable Bessel polynomial of degree \(n\) is defined as:

\[
y_n(x) \triangleq \sum_{k=0}^{n} \frac{(n + k)!}{(n - k)!k!} \left(\frac{x}{2}\right)^k.
\]

The following example shows that there exists multiplier sequences that maps some stable polynomial to a stable polynomial of the Jacobi class, explicitly, to the class of stable Bessel polynomials. By using 2nd and 3rd degree Bessel polynomials we illustrate the relation between stable polynomials and stable polynomials of the Jacobi class by the use of a multiplier sequence of the decreasing kind.

The sequence \(L = \{h(k)\}_{k=0}^{\infty}\) is a CZDS by means of Lemma 4.3 if \(h(s)\) is such that \(h(0) \neq 0\) and all roots are real and negative; let \(B(s)\) be a Bessel polynomial computed from Definition 4.8. Then for a Hurwitz polynomial \(p(s) = \sum_{k=0}^{n} a_k s^k\), we obtain the following polynomial

\[
L[p(s)] = \sum_{k=0}^{n} h(k)a_k s^k = B(s).
\]

**Example 4.9.** We begin by obtaining the Bessel polynomial of second degree, from Definition 4.8, i.e.:

\[
B(x) = \frac{3}{2}x^2 + 3x + 1
\]

then, finding a polynomial which roots are real and negative we develop the sequence,

\[
h(x) = x^2 + 4x + 2
\]
with \( h(0) = 2, \ h(1) = 7, \ h(2) = 14, \) then
\[
\frac{3}{2}x^2 + 3x + 1 = h(2)a_2x^2 + h(1)a_1x + (0)a_0
\]
from this we get the following stable polynomial:
\[
p(x) = \frac{3}{28}x^2 + \frac{3}{7}x + \frac{1}{2}
\]
Using a Bessel polynomial of third degree:
\[
B(x) = 15x^3 + 15x^2 + 6x + 1
\]
and the following polynomial with real and negative roots:
\[
h(x) = x^3 + x^2 + 2x + 1
\]
with \( h(0) = 1, \ h(1) = 5, \ h(2) = 17, \ h(3) = 43, \) then
\[
15x^3 + 15x^2 + 6x + 1 = h(3)a_3x^3 + h(2)a_2x^2 + h(1)a_1x + h(0)a_0
\]
we find the following stable polynomial:
\[
p(x) = \frac{15}{43}x^3 + \frac{15}{17}x^2 + \frac{6}{5}x + 1.
\]

Throughout the study of multiplier sequences we have come to realize the following conjecture.

**Conjecture 4.10.** Given a class of stable Jacobi polynomials (Bessel, Legendre, first and second kind Chebyshev and Gegenbauer [3]), there exists a multiplicative sequence that maps some stable polynomial into another stable Jacobi polynomial.

## 5 Operators and products preserving passivity

**Definition 5.1.** The diamond product of two real polynomials \( f(s) \) and \( g(s) \) is the polynomial
\[
(f \circ g)(s) \triangleq \sum_{k \geq 0} \frac{f^{(k)}(s)g^{(k)}(s)}{k!k!} s^k (s + 1)^k
\]
with \( f^{(k)}(s) \triangleq \frac{d^k f(s)}{ds^k} \) and \( g^{(k)}(s) \triangleq \frac{d^k g(s)}{ds^k} \).
Notice that the diamond product is bilinear, i.e., the product \( f \circ g \) is linear in the first and second components.

**Definition 5.2.** The Wagner product of two real polynomials \( f(s) \) and \( g(s) \) is the polynomial

\[
(f \circ g)(s) \triangleq \sum_{k \geq 0} \frac{[(1-s)^{-n_f} f(s)]^{(k)} [(1+s)^{-n_g} g(s)]^{(k)}}{k!k!} s^k (1-s)^{d-2k}
\]

where \( n_f \triangleq \deg f(s) \), \( n_g \triangleq \deg g(s) \) and

\[
d = \deg \sum_{k \geq 0} \frac{[(1+s)^{n_f} f\left(\frac{s}{1+s}\right)]^{(k)} [(1+s)^{n_g} g\left(\frac{s}{1+s}\right)]^{(k)}}{k!k!} s^k (1+s)^k.
\]

The Wagner product is also bilinear as the diamond product.

**Lemma 5.3 ([2]).** If \( f(s) \) and \( g(s) \) are real polynomials with all their roots in the interval \([-1, 0]\), then the diamond product \( (f \circ g)(s) \) has all its roots in the same interval.

**Lemma 5.4.** If the polynomials \( f(s) \) and \( g(s) \) are polynomials with all their roots in the interval \((-\infty, 0)\), then the Wagner product \( (f \circ g)(s) \) has all its roots in the same interval, i.e., the Wagner product of stable real-rooted polynomials is again a stable real-rooted polynomial.

**Proof.** First, we recall that a standard polynomial \( p(s) \) of degree \( n_p \) has all their roots in the interval \((-\infty, 0)\) if and only if the polynomial

\[
\hat{p}(s) = (1+s)^{n_p} p\left(\frac{s}{1+s}\right)
\]

has all its roots in the interval \([-1, 0]\), and a standard polynomial \( q(s) \) of degree \( n_q \) has all their roots in the interval \([-1, 0]\) if and only if the polynomial

\[
\hat{q}(s) = (1-s)^{n_q} q\left(\frac{s}{1-s}\right)
\]

has all its roots in the interval \((-\infty, 0)\).

Second, the diamond product of the \( f(s) \) and \( g(s) \) polynomials with all their roots in the interval \((-\infty, 0)\) can be written as

\[
(\hat{f} \circ \hat{g})(s) = \sum_{k \geq 0} \frac{\hat{f}^{(k)}(s)\hat{g}^{(k)}(s)}{k!k!} s^k (s+1)^k
\]

\[
= \sum_{k \geq 0} \frac{[(1+s)^{n_f} f\left(\frac{s}{1+s}\right)]^{(k)} [(1+s)^{n_g} g\left(\frac{s}{1+s}\right)]^{(k)}}{k!k!} s^k (1+s)^k.
\]
Third, observe that by Lemma 5.3, \((\hat{f} \circ \hat{g})(s)\) has all its roots in the interval \([-1, 0]\). Therefore
\[
(\hat{f} \circ \hat{g})(s) = (f \odot g)(s)
\]
has all its roots in the interval \((-\infty, 0]\). \hfill \Box

**Definition 5.5.** Let \(f\) and \(g\) be real polynomials. We say that \(f\) and \(g\) alternate if \(f\) and \(g\) are real-rooted and either of the following conditions hold:

1. \(\text{deg}(g) = \text{deg}(f) = d\) and
\[
\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \leq \beta_{d-1} \leq \alpha_d \leq \beta_d
\]
where \(\alpha_1 \leq \cdots \leq \alpha_d\) and \(\beta_1 \leq \cdots \leq \beta_d\) are the zeros of \(f\) and \(g\) respectively.

2. \(\text{deg}(f) = \text{deg}(g) + 1 = d\) and
\[
\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \leq \beta_{d-1} \leq \alpha_d
\]
where \(\alpha_1 \leq \cdots \leq \alpha_d\) and \(\beta_1 \leq \cdots \leq \beta_{d-1}\) are the zeros of \(f\) and \(g\) respectively. If all the inequalities above are strict then \(f\) and \(g\) are said to strictly alternate.

Moreover, if \(f\) and \(g\) are as in case 2 then we say that \(g\) interlaces \(f\). Case 1 or Case 2 are denoted by \(g \lesssim f\). In the strict case we write \(g \prec f\). If the leading coefficient of \(f\) is positive we say that \(f\) is standard.

**Lemma 5.6 ([2]).** Let \(h(s)\) be a real polynomial with all its roots in the interval \((-1, 0)\) and let \(f(s)\) be a polynomials with all roots real numbers. Then

1. If \(h \circ f\) all its roots are real numbers, and if \(g \lesssim f\) then
\[
h \circ g \lesssim h \circ f
\]

2. If \(h\) has all its roots in the interval \((-1, 0)\) and it is simple-rooted and \(f\) is simple-rooted then \(h \circ f\) is simple-rooted and
\[
h \circ g \prec h \circ f
\]

for all \(g \prec f\).

**Definition 5.7.** Let \(f(s) = f_e(s^2) + sf_o(s^2)\) be the decomposition of the real polynomial \(f\) in its even part \(f_e(s)\) and its odd part \(f_o(s)\). Define the linear operator \(\overline{\circ}_h : \mathbb{R}[s] \to \mathbb{R}[s]\) as
\[
\overline{\circ}_h(f(s)) \triangleq (h \circ f_e)(s^2) + s(h \circ f_o)(s^2).
\]
Notice that \(\overline{\circ}_h(f(s)) \neq h \circ (f_e(s^2) + sf_o(s^2))\). However, \(\overline{\circ}_h\) is a linear operator (see the proof of Theorem 5.12 below).
Lemma 5.8 ([12]). Let \( f(s) = f_e(s^2) + sf_o(s^2) \) be a real polynomial, then \( f(s) \) is Hurwitz stable if and only if \( f_e(s) \) and \( f_o(s) \) are standard, have only negative real roots, and \( f_o < f_e \).

Theorem 5.9. If \( h(s) \) has all its roots in the interval \((−∞, 0)\), it is simple-rooted, and \( f \) is a Hurwitz stable polynomial, then \( \overline{\circ}_h(f(s)) \) is a Hurwitz stable polynomial.

We say that a polynomial \( f(s) \) is simple-rooted, if all roots are different real numbers.

Proof. First, notice that by Lemma 5.8, \( f_e(s) \) and \( f_o(s) \) are standard, have only negative real roots, and \( f_o < f_e \).

Now, by Lemma 5.4 \( (h \circ f_e)(s) \) and \( (h \circ f_o)(s) \) have all their roots in the interval \((−∞, 0)\) i.e., have only negative real roots.

Second, notice that \( f_o < f_e \Leftrightarrow \tilde{f}_o < \tilde{f}_e \Leftrightarrow \tilde{f}_o < \tilde{f}_e \). Then using the Lemma 5.6 Item 2, \( (h \circ f_e)(s) \) and \( (h \circ f_o)(s) \) are simple-rooted and \( h \circ f_o < h \circ f_e \), and it is clear that \( (h \circ f_e)(s) \) and \( (h \circ f_o)(s) \) are standard, by linearity of the operator \( (h \circ _) \). In consequence, again by Lemma 5.8, \( \overline{\circ}_h(f(s)) \) is a Hurwitz stable polynomial. \( \square \)

Definition 5.10. The circle-point product of two standard real polynomials

\[
\begin{aligned}
f(s) &= f_n s^n + \cdots + f_0 \\
g(s) &= g_n s^n + \cdots + g_0
\end{aligned}
\]

is the polynomial

\[
(f \circ g)(s) \triangleq \sum_{k=0}^{n} k! f_k g_k s^k.
\]

Proposition 5.11. If the real polynomial \( h(s) \) has all its roots in the interval \((−∞, 0)\), and \( f \) is a Hurwitz stable polynomial, then \( \overline{\circ}_h(f(s)) \) is a Hurwitz stable polynomial. Where the linear operator \( \overline{\circ}_h : \mathbb{R}[s] → \mathbb{R}[s] \) is defined as

\[
\overline{\circ}_h (f(s)) \triangleq (h \circ f_e)(s^2) + s (h \circ f_o)(s^2).
\]

Proof. The proof is similar to Theorem 5.9, and it use the Lemma 10 and the Theorem 12 in [12] and the Lemma 5.8. \( \square \)

The following Theorem is a generalization of the previous results.

Theorem 5.12. If \( φ : \mathbb{R}[s] → \mathbb{R}[s] \) is a linear operator preserving simple-rootedness and negative real roots. Then the operator

\[
\overline{\circ} : \mathbb{R}[s] → \mathbb{R}[s]
\]
defined as
\[
\overline{\phi}(f(s)) \triangleq \phi(f_e)(s^2) + s\phi(f_o)(s^2)
\]
preserves Hurwitz stable polynomials and it is linear.

Proof. Let \( f(s) = f_e(s^2) + sf_o(s^2) \) be a Hurwitz stable polynomial. Now, by Theorem 9 in [2], if \( f_o > f_e \), then \( \phi(f_o) > \phi(f_e) \), and by hypothesis \( \phi(f_o) \) and \( \phi(f_e) \) have only negative real roots. By linearity \( \phi(f_o) \) and \( \phi(f_e) \) are standard. Therefore by Lemma 5.8, the polynomial \( \overline{\phi}(f(s)) \) is Hurwitz stable.

Let \( f(s) = f_e(s^2) + sf_o(s^2) \) and \( g(s) = g_e(s^2) + sg_o(s^2) \) be Hurwitz stable polynomials, and \( a, b \in \mathbb{R} \). For the prove of linearity, consider the following: By decomposition of \( f(s) \) and \( g(s) \)
\[
\overline{\phi}(af(s) + bg(s)) = \overline{\phi}(af_e(s^2) + bg_e(s^2) + s(af_o(s^2) + bg_o(s^2)))
\]
then by definition of \( \overline{\phi} \)
\[
\overline{\phi}(af(s) + bg(s)) = \overline{\phi}(af_e + bg_e)(s^2) + s\overline{\phi}(af_o + bg_o)(s^2)
\]
by linearity of \( \phi \)
\[
\overline{\phi}(af(s) + bg(s)) = (a\phi(f_e) + b\phi(g_e))(s^2) + s(a\phi(f_o) + b\phi(g_o))(s^2)
\]
by associativity of \( \phi \)
\[
\overline{\phi}(af(s) + bg(s)) = a(\phi(f_e) + s\phi(f_o))(s^2) + b(\phi(g_e) + s\phi(g_o))(s^2)
\]
by the definition of \( \overline{\phi} \)
\[
\overline{\phi}(af(s) + bg(s)) = a\overline{\phi}(f(s)) + b\overline{\phi}(g(s))
\]
\[\square\]

Lemma 5.13 ([10]). Let \( q(s) = f(s) + jg(s) \) a complex polynomial, where \( f(s) \) and \( g(s) \) are real polynomials. Then, all roots of \( q(s) \) are in the open left half-plane (i.e., it is a complex Hurwitz stable polynomial) if and only if the roots of \( f_e(-s^2) - sg_o(-s^2) \) and \( g_e(-s^2) + sf_o(-s^2) \) are simple, real and separate each other (i.e., the roots of the last two polynomials alternate or interlace).

Proof. The Lemma is consequence of the Theorem 9.1.7 in [10], and the substitution of \( s \) by \( js \).
\[\square\]

A generalization from Theorem 5.12 to the case of complex Hurwitz stable polynomials as well as to SPR0 functions is the following result.
Theorem 5.14. If \( \phi : \mathbb{R}[s] \to \mathbb{R}[s] \) is a linear operator preserving simple-rootedness and negative real roots. Then the operator
\[
\overline{\phi} : \mathbb{R}[s] \to \mathbb{R}[s]
\]
defined as
\[
\overline{\phi}(f(s)) = \phi(f_0)(s^2) + s\phi(f_0)(s^2).
\]
1. Preserves complex Hurwitz stable polynomials i.e., if \( b(s) = f(s) + jg(s) \) is a complex Hurwitz stable polynomial, then the complex polynomial
\[
\overline{\phi}(b(s)) = \phi(f_e)(s^2) + s\phi(f_0)(s^2) + j\left(\phi(g_e)(s^2) + s\phi(g_0)(s^2)\right)
\]
is Hurwitz stable.
2. Preserves SPR0 functions i.e., if \( \frac{b(s)}{q(s)} \) is a SPR0 function, then \( \frac{\overline{\phi}(b(s))}{\overline{\phi}(q(s))} \) is a SPR0 function.

Proof.

1. Consider the complex Hurwitz stable polynomial \( b(s) = f(s) + jg(s) \), notice that \( b(s) = f_e(s^2) + sf_0(s^2) + j(g_e(s^2) + sg_0(s^2)) \). Applying the operator \( \overline{\phi} \) to \( q(s) \), we obtain
\[
\overline{\phi}(b(s)) = \phi(f_e)(s^2) + s\phi(f_0)(s^2) + j\left(\phi(g_e)(s^2) + s\phi(g_0)(s^2)\right).
\]
Now by the last Lemma the complex polynomial \( \overline{\phi}(q(s)) \) is Hurwitz stable if and only if the roots of \( \phi(f_e)(-s^2) - s\phi(g_0)(-s^2) \) and \( \phi(g_e)(-s^2) + s\phi(f_0)(-s^2) \) are simple, real and separate each other. To prove the last statement, first observe that
\[
b(js) = f_e(-s^2) - sg_0(-s^2) + j(g_e(-s^2) + sf_0(-s^2))
\]
and
\[
\overline{\phi}(b(js)) = \phi(f_e)(-s^2) - s\phi(g_0)(-s^2) + j\left(\phi(g_e)(-s^2) + s\phi(f_0)(-s^2)\right).
\]
On the other side, notice that
\[
\overline{\phi}(b(\sigma))_{\sigma=js} = \phi(f_e)(-s^2) - s\phi(g_0)(-s^2) + j\left(\phi(g_e)(-s^2) + s\phi(f_0)(-s^2)\right).
\]
Therefore, considering the hypothesis of Hurwitz stability of \( q(s) \) and Lemma 5.13, the roots of the polynomials \( f_e(s^2) + sf_0(s^2) \) and \( g_e(s^2) + sg_0(s^2) \) are simple, real and separate each other, and that the operator \( \phi \) preserves simple-rootedness and negative real roots. Then, again by Lemma 5.13 the roots of \( \phi(f_e)(-s^2) - s\phi(g_0)(-s^2) \) and \( \phi(g_e)(-s^2) + s\phi(f_0)(-s^2) \) are simple, real and separate each other. Therefore, the complex polynomial \( \overline{\phi}(b(s)) \) is Hurwitz stable.
2. The proof of this item is based on Lemma 2.6 and Theorem 5.12 and item 1 of Theorem 5.14.

\[ \Box \]

Notice that with linear operators preserving Hurwitz stable polynomials is possible to build new classes of products of Hurwitz stable polynomials. For example, let \( f(s) \) and \( g(s) \) be Hurwitz stable polynomials, and \( \phi, \varphi : \mathbb{R}[s] \to \mathbb{R}[s] \) linear operators preserving Hurwitz stable polynomials. Define the product \( \circ_{\phi, \varphi} \) of Hurwitz stable polynomials as

\[ f(s) \circ_{\phi, \varphi} g(s) \triangleq \overline{\phi(f(s))} \circ \overline{\varphi(g(s))}. \]

The product \( \bullet_{\phi, \varphi} \) of Hurwitz stable polynomials as

\[ f(s) \bullet_{\phi, \varphi} g(s) \triangleq \overline{\phi(f(s))} \overline{\varphi(g(s))}. \]

The products \( \circ_{\phi, \varphi} \) and \( \bullet_{\phi, \varphi} \) are linear in each component, associative and preserve Hurwitz stability, but they are not commutative.

Continuing this section and based in a surprising result of [14], it is possible extend the Theorem 5.12 to families of polynomials depending analytically on parameters (i.e. the coefficients are analytic functions of one parameter).

**Lemma 5.15 ([14]).** Consider a polynomial

\[ p(x, s) = s^d + \sum_{i=1}^{d} a_i(x) s^{d-i} \]

with \( a_i(x) \) real analytic functions in an open interval \( I \subset \mathbb{R} \). Assume that for each \( x \in I \) all the roots of the polynomial \( s \to p(x, s) \) are real. Then there exist real analytic functions \( r_i : I \to \mathbb{R} \) such that

\[ p(x, s) = \prod_{i=1}^{d} (s - r_i(x)), \ x \in I, \ s \in \mathbb{R}. \]

Now consider a family of polynomials

\[ q(x, s) = b_d s^d + \sum_{i=1}^{d} b_i(x) s^{d-i} \quad (1) \]

where \( b_d \in \mathbb{R}^+ - \{0\} \) and the following decomposition

\[ q(x, s) = q_e(x, s^2) + sq_o(x, s^2) \]
with

\[ q_e(x, s) = \tilde{b}_{n_e} s^{n_e} + \sum_{i=1}^{n_e} \tilde{b}_i(x) s^{d-i} = \prod_{i=1}^{n_e} (s - \tilde{r}_i(x)) \]
\[ q_o(x, s) = \hat{b}_{n_o} s^{n_o} + \sum_{i=1}^{n_o} \hat{b}_i(x) s^{d-i} = \prod_{i=1}^{n_o} (s - \hat{r}_i(x)) \]

where \( \tilde{b}_{n_e}, \hat{b}_{n_o} \in \mathbb{R}^+ - \{0\} \), \( x \in I, s \in \mathbb{R} \), \( \tilde{r}_i, \hat{r}_i : I \rightarrow \mathbb{R} \) are real analytic functions, and once of the following conditions is satisfied:

1. If \( \deg(q_o) = \deg(q_e) = n \) then
   \[ \hat{r}_1(x) \leq \tilde{r}_1(x) \leq \tilde{r}_2(x) \leq \cdots \leq \tilde{r}_{n-1}(x) \leq \tilde{r}_n(x) \leq \tilde{r}(x) \]
   where \( \hat{r}_1(x) \leq \cdots \leq \hat{r}_n(x) \) and \( \tilde{r}_1(x) \leq \cdots \leq \tilde{r}_n(x) \) are the zeros of \( q_e \) and \( q_o \) respectively for all \( x \in I, s \in \mathbb{R} \).

2. If \( \deg(q_e) = \deg(q_o) + 1 = d \) then
   \[ \hat{r}_1(x) \leq \tilde{r}_1(x) \leq \tilde{r}_2(x) \leq \cdots \leq \tilde{r}_{n-1}(x) \leq \tilde{r}_n(x) \]
   where \( \hat{r}_1(x) \leq \cdots \leq \hat{r}_n(x) \) and \( \tilde{r}_1(x) \leq \cdots \leq \tilde{r}_{n-1}(x) \) are the zeros of \( q_e \) and \( q_o \) respectively for all \( x \in I, s \in \mathbb{R} \). We denote by \( q_o < q_e \) the condition 1 or 2.

Denote by \( \mathbb{R}[x][s] \) the set of polynomials \( q(x, s) \) with coefficients real analytic functions in an open interval \( I \subset \mathbb{R} \), for all \( x \in I \).

A Hurwitz stable polynomial family \( q(x, s) \) is a set of polynomials, such that for each \( x_0 \in I \subset \mathbb{R} \) the polynomial \( q(x_0, s) \) is a Hurwitz stable polynomial in \( \mathbb{R}[s] \).

**Corollary 5.16.** If \( \phi : \mathbb{R}[x][s] \rightarrow \mathbb{R}[x][s] \) is a linear operator preserving simple-rootedness and negative real roots. Suppose that the polynomial family \( q(x, s) \), as in (1), has a decomposition \( q_e(x, s^2) + sq_o(x, s^2) \) with \( \tilde{b}_{n_e}, \hat{b}_{n_o} \in \mathbb{R}^+ - \{0\} \), \( x \in I, s \in \mathbb{R} \), the coefficients of \( q(x, s) \) are real analytic functions in an open interval \( I \subset \mathbb{R} \), also \( \hat{r}_i(x), \tilde{r}_i(x) \in \mathbb{R}^+ - \{0\} \) and condition 1 or 2 are satisfy. Then the operator \( \overline{\phi} : \mathbb{R}[x][s] \rightarrow \mathbb{R}[x][s] \) defined as

\[ \overline{\phi}(q(x, s)) = \phi(q_e(x, -))(s^2) + s\phi(q_o(x, -))(s^2) \]

preserves Hurwitz stable polynomial families and it is linear.

**Proof.** It is consequence of the Lemma 5.15 and the Theorem 5.12.

Also, it is possible to generalize the Theorem 5.14 in a similar form as Corollary 5.16. It is interesting to observe that the Corollary 5.16 can be understood in the sense of preservation of robust parametric stability. Recently
in [13], it has been proved a generalization of the Lemma 5.15 to the multi-parametric real case i.e., for \( x \in U \subset \mathbb{R}^n \), Using this results the Corollary 5.16 can be generalized for robust multi-parametric stability.

Finally, in this part the operators of deepness \( k \) are introduced, to give the last generalization of the theorems.

Consider the following recursive decomposition of the polynomial \( f(s) \):

<table>
<thead>
<tr>
<th>Recursive polynomial decomposition</th>
<th>Components</th>
<th>Deepness</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_k(s^2) + sf_0(s^2) )</td>
<td></td>
<td>2^1</td>
</tr>
<tr>
<td>( (f_{ee}(s^2) + sf_{eoe}(s^2))(s^2) + \cdots )</td>
<td></td>
<td>2^2</td>
</tr>
<tr>
<td>( s \left[ (f_{oe}(s^2) + sf_{oo}(s^2))(s^2) \right] )</td>
<td></td>
<td>2^3</td>
</tr>
<tr>
<td>( (f_{eoe}(s^2) + sf_{eoo}(s^2))(s^2) + \cdots )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s \left[ (f_{ee}(s^2) + sf_{eoe}(s^2))(s^2) + sf_{eoo}(s^2) \right] )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( (2^k)f(s) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is clear that the deepness \( k \) is bounded by the order \( n \) of the polynomial \( f(s) \) i.e., \( k \leq n \).

**Definition 5.17.** If \( \phi : \mathbb{R}[s] \rightarrow \mathbb{R}[s] \) is a linear operator, a linear operator \( \overline{\phi}_k \) of deepness \( k \) on the polynomial \( f(s) \) is a linear operator with action in the components of \( k \) deepness of \( (2^k)f(s) \), only.

For example \( \overline{\phi}_3 \) is given by

\[
\overline{\phi}_3 \{ \left\{ (f_{ee}(s^2) + sf_{eoe}(s^2))(s^2) + s \left( f_{eoe}(s^2) + sf_{eoo}(s^2) \right)(s^2) \right\} (s^2) + \cdots \} = \cdots
\]

In this example the bicomponents of 3 deepness of \( f(s) \) are \( f_{ee}(s^2) + sf_{eoe}(s^2) \), \( f_{eoe}(s^2) + sf_{eoo}(s^2) \), \( f_{ee}(s^2) + sf_{eoe}(s^2) \), \( f_{eoe}(s^2) + sf_{eoo}(s^2) \), \( f_{eoe}(s^2) + sf_{eoo}(s^2) \), and \( f_{eoe}(s^2) + sf_{eoo}(s^2) \).

Now is possible to present the result.

**Corollary 5.18.** If \( \phi : \mathbb{R}[s] \rightarrow \mathbb{R}[s] \) is a linear operator preserving simple-rootedness and negative real roots. Then the operator \( \overline{\phi}_k : \mathbb{R}[s] \rightarrow \mathbb{R}[s] \), as defined in Definition 5.17, preserves Hurwitz stable polynomials and it is linear.

**Proof.** By Lemma 5.8 each one of the bicomponents of \( k \) deepness \( f_k(s^2) + sf_k(s^2) \) is an Hurwitz stable polynomial. In consequence by Theorem 5.12,
all the bicomponents of $k$ deepness $\varphi_k(f_k)(s^2) + s\varphi_k(f_k)(s^2)$ of the resultant polynomial are Hurwitz stable and by Lemma 5.8 again, the polynomial $\varphi_k(f)(s)$ is an Hurwitz stable polynomial. The prove of linearity is similar to the Theorem 5.12.

\[\square\]

**Remark 5.19.** The Gauss-Lucas Theorem states that if a polynomial $p(s)$ has its zeros contained in some given convex set $K$, then its derivative $\frac{dp(s)}{ds}$ has all its zeros in $K$ as well (unless $p(s)$ is constant). In particular, if all the zeros are real, then so are the zeros of the derivative. For instance, if all zeros of $p(s)$ are real and negative, then so are the zeros of the derivative.

In the following example we present the decomposition of deepness $k$ for a polynomial using the standard derivative operator.

**Example 5.20.** Let start with a Hurwitz polynomial

\[p(s) = (s + 1)^4 (s + 2)^3 (s + 3)\]
\[= s^8 + 13s^7 + 72s^6 + 222s^5 + 417s^4 + 489s^3 + 350s^2 + 140s + 24\]

and the operator $\varphi_k$ is constructed by the standard derivative operator. The even ($p_e$) and odd ($p_o$) decomposition for the polynomial are given by

\[p_e(s^2) = s^8 + 72s^6 + 417s^4 + 350s^2 + 24\]
\[= (s^2)^4 + 72(s^2)^3 + 417(s^2)^2 + 350(s^2) + 24\]
\[p_e(s) = s^4 + 72s^3 + 417s^2 + 350s + 24\] (2)

\[sp_o(s^2) = s [13s^6 + 222s^4 + 489s^2 + 140]\]
\[= 13(s^2)^3 + 222(s^2)^2 + 489(s^2) + 140\]
\[p_o(s) = 13s^3 + 222s^2 + 489s + 140\] (3)

Applying the derivative operator to $p_e$ in (2) and $p_o$ in (3)

\[\frac{p_e(s)}{ds} = 4s^3 + 216s^2 + 834s + 350\]
\[\frac{p_o(s)}{ds} = 39s^2 + 444s + 489\]

Constructing the polynomial of deepness 1, $f_1(s)$

\[f_1(s) = \left.\frac{p_e}{ds}\right|_{s^2} + s\left.\frac{p_o}{ds}\right|_{s^2}\]
\[= [4s^3 + 216s^2 + 834s + 350]_{s^2} + s \left[39s^2 + 444s + 489\right]_{s^2}\]
\[f_1(s) = 4s^6 + 216s^5 + 834s^4 + 350 + 39s^5 + 444s^3 + 489s\]
which is a Hurwitz polynomial by Corollary 5.18.

For the decomposition of deepness 2

\[ p_{oo} (s^2) = s^4 + 417s^2 + 24 = (s^2)^2 + 417 (s^2) + 24 \]

\[ p_{oo}(s) = s^2 + 417s + 24 \]

\[ sp_{oo} (s^2) = s [72s^2 + 350] = s [72 (s^2) + 350] \]

\[ p_{oe}(s) = 72s + 350 \]

\[ p_{oe} (s^2) = 222s^2 + 140 = 222 (s^2) + 140 \]

\[ p_{oe}(s) = 222s + 140 \]

\[ sp_{oo} (s^2) = s [13s^2 + 489] = s [13 (s^2) + 489] \]

\[ sp_{oo}(s) = 13s + 489 \]

Construction the polynomial of deepness 2, \( f_2(s) \)

\[
\begin{align*}
    f_2(s) &= \left. \frac{p_{oe}}{ds} \right|_{s^2} + s \left. \frac{p_{eo}}{ds} \right|_{s^2} + s \left. \frac{p_{oe}}{ds} \right|_{s^2} + s \left. \frac{p_{oo}}{ds} \right|_{s^2} \\
    &= \left. (2s + 417) \right|_{s^2} + s \left. (72) \right|_{s^2} + s \left. (222) \right|_{s^2} + s \left. (13) \right|_{s^2} \\
    f_2(s) &= \left. [2s^2 + 72s + 417 + 222 + 13s] \right|_{s^2} \\
    f_2(s) &= 2s^2 + 13s^3 + 72s^2 + 222s + 417
\end{align*}
\]

which is also a Hurwitz polynomial by Corollary 5.18. \qed

6 Conclusions

In this paper we have developed new methods for generating SSPM’s and families of such linear operators. We believe that these results provide an insight into the theory of matrix stability preserving maps. This can also be extended in several directions. A possible application of the results presented in this paper is in robust stability theory.

On the other hand we generalized a result presented by in [1], due to Talbot, when preserving the SPR0 property, using different linear operators in the numerator and the denominator.

The theory of distributions of zeros of polynomials, it is useful to show, how by the use of multiplier sequences, a stable polynomial could be mapped into stable polynomials of the Jacobi class, and then be represented as SSPM operators.

Using generalized products of polynomial operators that preserve simple-rootness and negative real roots are built to preserve stability and SPR0 functions. Extensions to families of polynomials depending on parameters and operators on deepness-\( k \) are given.

As far as we know, it is an open question whether this can be extended to cover non-linear operators, or of other types.
References


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