

Associated Laguerre Polynomials: Monomiality and Bi-Orthogonal Functions

G. Dattoli ¹ and M. Migliorati ²

ENEA, Unità Tecnico Scientifica Tecnologie Fisiche Avanzate
Centro Ricerche Frascati, C.P. 65, 00044 Frascati, Rome, Italy

Abstract

We exploit the concepts and the formalism associated with the principle of monomiality to derive the orthogonality properties of the associated Laguerre polynomials. We extend a recently developed technique of algebraic nature and comment on the usefulness of the proposed method.

Keywords: Monomiality, Bi-orthogonal functions, Special polynomials, Laguerre polynomials, Jacobi polynomials

1 INTRODUCTION

It has been shown in refs. (1) that the concepts associated with the monomiality treatment of classical and generalized polynomials² can be exploited to derive a unified treatment of the orthogonality properties of families of polynomials, used in pure and applied mathematics. The method has so far been applied to various families of Hermite (including higher orders and multi-index cases) and Laguerre polynomials. In this paper we will extend the method to Associated Laguerre and touch on the Jacobi family.

To this aim we remind that, according to ref. (2), a polynomial $p_n(x)$ ($n \in N, x \in C$) is said a quasi-monomial, whenever two operators \widehat{M}, \widehat{P} , called

¹dattoli@frascati.enea.it

²Dipartimento di Energetica, Università di Roma "La Sapienza" Via A. Scarpa, 14 - 00161 Rome, Italy

multiplicative and derivative operators respectively, can be defined in such a way that

$$\begin{aligned}\widehat{P}p_n(x) &= np_{n-1}(x), \\ \widehat{M}p_n(x) &= p_{n+1}(x)\end{aligned}\quad (1)$$

The properties of the $p_n(x)$ polynomials are contained in the explicit definition of the relevant \widehat{M}, \widehat{P} operators. If $p_0(x) = 1$, as often happens, we can write the polynomials and their generating function as

$$\begin{aligned}p_n(x) &= \widehat{M}^n(1), \\ \sum_{n=0}^{\infty} \frac{t^n}{n!} p_n(x) &= \exp(t\widehat{M})(1)\end{aligned}\quad (2)$$

while the relevant eigenvalue problem reads

$$\widehat{M}\widehat{P}p_n(x) = np_n(x) \quad (3)$$

The previous formalism has also been exploited in Ref. (1) to investigate the properties of function families $g_n(x)$, bi-orthogonal to $p_n(x)$ and specified according to the operational definition

$$g_n(x) = \frac{1}{\left(\widehat{M}^+\right)^{n+1}}(1) \quad (4)$$

where \widehat{M}^+ denotes the hermitian conjugate of the derivative operator. The recurrence properties of the $g_n(x)$ in terms of the multiplicative and derivative operators can be written as

$$\begin{aligned}\widehat{M}^+g_n(x) &= g_{n-1}(x), \\ \widehat{P}^+g_n(x) &= (n+1)g_{n+1}(x)\end{aligned}\quad (5)$$

which can be combined to get the identity

$$\widehat{P}^+\widehat{M}^+g_n(x) = ng_n(x) \quad (6)$$

We can furthermore use the relation

$$a^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^{\infty} \exp(-sa) s^{\nu-1} ds \quad (7)$$

along with eq. (4), to define the functions $g_n(x)$ according to the integral representation

$$g_n(x) = \frac{1}{n!} \int_0^\infty s^n \exp(-s\widehat{M}^+) ds(1) \quad . \quad (8)$$

The previously described procedure has been exploited to develop a unified point of view on the orthogonality properties of Hermite and Laguerre polynomials, including, in the case of the Hermite family, also the multi-index and higher order extensions (Ref. (1)).

In this paper, as already remarked, we will extend the method to the associated Laguerre, Legendre and Jacobi polynomials, therefore, before getting into the main body of the paper, we recall a few notions necessary for the understanding of what follows.

In the case of the Laguerre polynomials the following realization of the \widehat{M}, \widehat{P} operators holds

$$\begin{aligned} \widehat{M} &= y - \widehat{D}_x^{-1} , \\ \widehat{P} &= -\frac{\partial}{\partial x} x \frac{\partial}{\partial x} \quad , \end{aligned} \quad (9)$$

where \widehat{D}_x^{-1} is the negative derivative operator defined in such a way that (ref. (3))

$$\widehat{D}_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_0^x (x - \xi)^{n-1} f(\xi) d\xi \quad , \quad (10)$$

thus getting for $f(x) = 1$

$$\widehat{D}_x^{-n} = \frac{x^n}{n!} \quad . \quad (11)$$

With these assumptions we find the polynomial family

$$L_n(x, y) = \left(y - \widehat{D}_x^{-1} \right)^n = n! \sum_{r=0}^n \frac{(-1)^r x^r y^{n-r}}{(n-r)! (r!)^2} \quad , \quad (12)$$

which reduce to the ordinary Laguerre for $y = 1$.

In the forthcoming sections we will see how the method can be conveniently extended to associated Laguerre and Jacobi polynomials.

2 MONOMIALITY AND THE ASSOCIATED LAGUERRE POLYNOMIALS

We will use the following realization of the \widehat{M}, \widehat{P} operators to extend the previously described technique to the associated Laguerre polynomials,

$$\begin{aligned}\widehat{M} &= y - \widehat{D}_{x,\nu}^{-1}, \\ \widehat{P} &= -\Gamma(\nu + 1) \left[\frac{\partial}{\partial x} x \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial x} \right]\end{aligned}\quad (13)$$

where the operator $\widehat{D}_{x,\nu}^{-1}$ is defined in such a way that its action on the unity is specified by

$$\widehat{D}_{x,\nu}^{-n}(1) = \frac{x^n}{\Gamma(n + \nu + 1)} \quad , \quad (14)$$

it is also easily proved that the above operators satisfy, as it must be, the commutation bracket

$$[\widehat{P}, \widehat{M}] = \widehat{1} \quad . \quad (15)$$

According to eq. (2) we define the polynomials

$$l_n^{(\nu)}(x, y) = (y - \widehat{D}_{x,\nu}^{-1})^n = n! \sum_{r=0}^n \frac{(-1)^r x^r y^{n-r}}{r! \Gamma(r + \nu + 1) (n-r)!} \quad , \quad (16)$$

which are linked to the ordinary associated Laguerre polynomials⁴ by the relation

$$L_n^{(\nu)}(x) = \frac{\Gamma(n + \nu + 1)}{n!} l_n^{(\nu)}(x) \quad . \quad (17)$$

It is worth stressing the following important identities

$$\begin{aligned}\left(-\frac{\partial}{\partial x} x \frac{\partial}{\partial x} - \nu \frac{\partial}{\partial x}\right) l_n^{(\nu)}(x, y) &= n l_{n-1}^{(\nu)}(x, y), \\ \frac{\partial}{\partial y} l_n^{(\nu)}(x, y) &= n l_{n-1}^{(\nu)}(x, y)\end{aligned}\quad , \quad (18)$$

which suggest that this family of polynomials satisfy the PDE

$$\begin{aligned}\frac{\partial}{\partial y} l_n^{(\nu)}(x, y) &= \left(-\frac{\partial}{\partial x} x \frac{\partial}{\partial x} - \nu \frac{\partial}{\partial x}\right) l_n^{(\nu)}(x, y), \\ l_n^{(\nu)}(x, 0) &= \frac{(-1)^n x^n}{\Gamma(n + \nu + 1)}\end{aligned}\quad . \quad (19)$$

We can now derive in a quite straightforward way some further properties of the polynomials defined in eq. (16).

The generating function can be obtained using the technique developed in ref. (2), which yields

$$\begin{aligned} \sum_{n=0}^{\infty} t^n l_n^{(\nu)}(x, y) &= \sum_{n=0}^{\infty} t^n (y - \widehat{D}_{x,\nu}^{-1})^n = \frac{1}{1-yt+t\widehat{D}_{x,\nu}^{-1}} = \frac{1}{1-yt} \sum_{s=0}^{\infty} \frac{(-1)^s t^s}{(1-yt)^s} \widehat{D}_{x,\nu}^{-s} = \\ &= \frac{1}{1-yt} E_{\nu}\left(\frac{xt}{1-yt}\right), E_{\nu}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{\Gamma(\nu+r+1)} \end{aligned} \tag{20}$$

thus finding for the ordinary case

$$\sum_{n=0}^{\infty} t^n \frac{n!}{\Gamma(\nu + n + 1)} L_n^{(\nu)}(x) = \frac{1}{1-t} E_{\nu}\left(\frac{xt}{1-t}\right) \tag{21}$$

In an analogous way we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} l_n^{(\nu)}(x, y) &= \exp(yt) \exp(-t\widehat{D}_{x,\nu}^{-1}) = \exp(yt) C_{\nu}(xt), \\ C_{\nu}(x) &= \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r! \Gamma(\nu+r+1)} \end{aligned} \tag{22}$$

thus also getting

$$\sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\nu + n + 1)} L_n^{(\nu)}(x) = \exp(t) C_{\nu}(xt) \tag{23}$$

We can now construct the bi-orthogonal functions associated with the $l_n^{(\nu)}(x)$, we define indeed the operator $\widehat{T}_{x,\nu}$ defined in such a way that

$$\left(\widehat{T}_{x,\nu}\right)^n (1) = \frac{(-1)^n x^{\nu+n}}{\Gamma(\nu + n + 1)} \tag{24}$$

which for $\nu = 0$ reduces to $(-1)^n \widehat{D}_x^{-n}$. In analogy to what we have done before we introduce the further operator

$$\widehat{M}^+ = y - \widehat{T}_{x,\nu} \tag{25}$$

and the use of the prescription given in the introductory remarks finally yields

$$\frac{1}{\left(\widehat{M}^+\right)^{\nu+n+1}} (1) = \frac{x^{\nu}}{\Gamma(n + \nu + 1)} \int_0^{\infty} \exp(-sy) s^{n+\nu} C_{\nu}(sx) ds \tag{26}$$

The use of the following relation between Tricomi and Bessel functions $J_{\nu}(x)$

$$C_{\nu}(x) = x^{-\frac{\nu}{2}} J_{\nu}(2\sqrt{x}) \tag{27}$$

allows to recast the r.h.s of eq. (26) as

$$\frac{x^\nu}{n!} \int_0^\infty \exp(-sy) s^{n+\nu} C_\nu(sx) ds = \frac{x^{\frac{\nu}{2}}}{n!} \int_0^\infty \exp(-sy) s^{n+\frac{\nu}{2}} J_\nu(2\sqrt{sx}) ds \quad . \quad (28)$$

By changing variable we find⁵

$$\begin{aligned} \frac{x^{\frac{\nu}{2}}}{n!} \int_0^\infty \exp(-sy) s^{n+\frac{\nu}{2}} J_\nu(2\sqrt{sx}) ds &= \Phi_m^{(\nu)}(x, y) = \frac{1}{y^{n+1}} \Lambda_n^{(\nu)}\left(\frac{x}{y}\right), \\ \Lambda_n^{(\nu)}(\eta) &= \frac{\eta^{\frac{\nu}{2}}}{n!} \int_0^\infty \exp(-\xi) \xi^{n+\frac{\nu}{2}} J_\nu(2\sqrt{\xi\eta}) d\xi = \exp(-\eta) \eta^\nu L_n^{(\nu)}(\eta) \end{aligned} \quad (29)$$

thus getting

$$\int_0^\infty l_n^{(\nu)}(x, y) \Phi_m^{(\nu)}(x, y) dx = \delta_{m,n} \quad , \quad (30)$$

which for $y = 1$ is immediately recognized as the ordinary orthogonality relation of the associated Laguerre polynomials (see Ref. (4)), the function defined in eq. (29) is bi-orthogonal to $l_n^{(\nu)}(x, y)$.

3 CONCLUDING REMARKS

According to the discussion of the previous section it is not difficult to realize that the polynomials $l_n^{(\nu)}(x, y)$ satisfy the operational equation

$$\begin{aligned} \frac{\partial}{\partial y} l_n^{(\nu)}(x, y) &= -\frac{\partial}{\partial \widehat{D}_{x,\nu}^{-1}} l_n^{(\nu)}(x, y), \\ l_n^{(\nu)}(x, 0) &= (-1)^n \widehat{D}_{x,\nu}^{-n} \end{aligned} \quad , \quad (31)$$

which can be exploited to get further insight into the properties of the associated Laguerre polynomials. The formal solution of eq. (31) reads indeed

$$l_n^{(\nu)}(x, y) = \exp\left(-y \frac{\partial}{\partial \widehat{D}_{x,\nu}^{-1}}\right) \left[(-1)^n \widehat{D}_{x,\nu}^{-n} \right] \quad . \quad (32)$$

Let us now consider the differential equation

$$\begin{aligned} \frac{\partial}{\partial y} F(x, y) &= \left(-\frac{\partial}{\partial x} x \frac{\partial}{\partial x} - \nu \frac{\partial}{\partial x}\right) F(x, y), \\ F(x, 0) &= f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} x^n \end{aligned}$$

It is evident that, according to the previous discussions we can write the relevant solution as

$$F(x, y) = \exp\left(y \left(-\frac{\partial}{\partial x} x \frac{\partial}{\partial x} - \nu \frac{\partial}{\partial x}\right)\right) f(x) = \sum_{n=0}^\infty (-1)^n f^{(n)}(0) L_n^{(\nu)}(x, y) \quad , \quad (33)$$

which, on account of eq. (23) can be written in terms of the following integral transform

$$F(x, y) = \exp\left(y\left(-\frac{\partial}{\partial x}x\frac{\partial}{\partial x}-\nu\frac{\partial}{\partial x}\right)\right)f(x) = e^{\frac{x}{y}} \int_0^{\infty} e^{-s} s^{\nu} C_{\nu}\left(\frac{x}{y}s\right) f(-ys) ds, y \neq 0 \quad (34)$$

Before concluding this paper let us consider an extension of the method to the case of the Jacobi polynomials.

To this aim we recall that the Legendre polynomials have been introduced in ref. (5) using the operational definition

$$R_n(x, y) = n!(\widehat{D}_y^{-1} - \widehat{D}_x^{-1})^n, \quad (35)$$

which, once expanded, yields

$$R_n(x, y) = (n!)^2 \sum_{r=0}^n \frac{(-1)^{n-r} x^{n-r} y^r}{((n-r)!)^2 (r!)^2}, \quad (36)$$

so that the ordinary Legendre are recovered as a particular case of the two variable polynomials given in eq. (36) as

$$P_n(x) = R_n\left(\frac{1-x}{2}, \frac{1+x}{2}\right). \quad (37)$$

We generalize the definition given in eq. (35) and introduce the polynomials

$$\begin{aligned} R_n^{(\mu, \nu)}(x, y) &= n!(\widehat{D}_{y, \nu}^{-1} - \widehat{D}_{x, \mu}^{-1})^n = \\ &= (n!)^2 \sum_{r=0}^n \frac{(-1)^{n-r} x^{n-r} y^r}{(n-r)! \Gamma(n-r+\mu+1) r! \Gamma(r+\nu+1)}. \end{aligned} \quad (38)$$

The Jacobi polynomials can be recognized as a particular case of (38). The use of the previously indicated methods allows a quite straightforward derivation of the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} R_n^{(\mu, \nu)}(x, y) = C_{\nu}(-yt) C_{\mu}(xt) = (-yt)^{-\frac{\nu}{2}} (xt)^{-\frac{\mu}{2}} J_{\nu}(2\sqrt{-yt}) J_{\mu}(2\sqrt{xt}). \quad (39)$$

It is evident that the method of monomiality associated with techniques of operational nature offers a powerful tool to investigate the properties of special functions, within a unified framework. The Authors believe that the same

method may provide an interesting, albeit unorthodox, mean to explore the theory of differential equations. We consider indeed the following example

$$\begin{aligned} \frac{\partial}{\partial x} x \frac{\partial}{\partial x} F(x, y) &= - \int_0^y F(x, \sigma) d\sigma, \\ F(0, y) &= g(y) \end{aligned} \quad (40)$$

whose solution can formally be written as follows

$$F(x, y) = C_0(x\widehat{D}_y^{-1})g(y) \quad (41)$$

which yields the Volterra series, written in Umbral Form

$$\begin{aligned} F(x, y) &= C_0(xG), \\ G^r &= \frac{1}{(v-1)!} \int_0^y (y-\sigma)^{r-1} g(\sigma) d\sigma \quad . \quad (35). \end{aligned}$$

In a forthcoming study we will investigate more deeply this aspect of the problem with particular reference to “time” ordering effects.

ACKNOWLEDGMENTS. The Authors express their sincere appreciation to Profs. P.E. Ricci and H.M. Srivastava for a number of clarifying and stimulating discussions.

REFERENCES

1. G. Dattoli, B. Germano and P. E. Ricci, *Appl. Math. And Comp.* **154**, 219, (2004).
2. G. Dattoli, H. M. Srivastava and K. Zhukovsky, *J. Comp. Math.* **182**, 165 (2005).
3. G. Dattoli, “Hermite-Bessel and Laguerre-Bessel functions a by-product of the monomiality principle” *Advanced special functions and applications* (Melfi, 1999) (D. Cocolicchio, G. Dattoli and H. M. Srivastava eds.) *Proc. Melfi Sch. Adv. Top. Math.*, Vol. 1, Rome 2000, pp. 147-164.
4. J. Kondo, *Integral equations*, Clarendon Press Oxford (1991).
5. L. C. Andrews, “Special functions for engineers and applied mathematicians” *Mc. Millan New York* (1985).

6. G. Dattoli, P. E. Ricci, and C. Cesarano, *Int. J. of non Linear Sciences and Numerical Simulations* **2**, 365 (2001).

Received: July 12, 2007