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Homotopy Perturbation Method Fokker-Planck Equation

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Abstract

In this article, we have used homotopy perturbation method (HPM) to solve the linear and nonlinear Fokker-Planck equations. To illustrate the capability and reliability of the method, some examples are provided. The results obtained using HPM are compared to the results of Adomian decomposition method (ADM) and then capability of these methods are also discussed.

Keywords: Fokker-Planck equation; Kolmogorov equation; Homotopy perturbation method; Adomian decomposition method

1 Introduction

The idea of HPM was first introduced by He in 1998 [1].By the homotopy technique in topology, a homotopy can be constructed with an embedding parameter $p \in [0, 1]$, which is considered as a small one. In this method, the solution is considered as the summation of a series which rapidly converge to the solution. HPM is a combination of the perturbation and homotopy methods. This method (HPM) can take the advantages of the conventional perturbation method while eliminating its restrictions. First author has employed HPM to solve nonlinear Schrodinger equation, Ganji applied HPM to study nonlinear heat transfer and porous media equations [3] and Rafei employed this technique to solve RLW and generalized modified Boussinesq equations

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[4]. In general, this method has been successfully applied to solve many types of linear and nonlinear equations in science and engineering by some authors [5-13]. The aim of this article is to employ HPM to solve of Fokker-Planck equation and comparing the results obtained with ADM [14].

2 Fokker-Planck equation

Fokker-Planck equation was first introduced by Fokker and Planck to describe the Brownian motion of particles [15]. This equation has been used in different fields in natural sciences such as quantum optics, solid-state physics, chemical physics, theoretical biology and circuit theory.
Fokker-Planck equation in general form can be expressed as follows [15]:

$$\frac{\partial u}{\partial t} = [-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)]u, \tag{1}$$

with the following initial condition:

$$u(x,0) = f(x), \qquad x \in R$$

where $u(x,y)$ is an unknown function. $A(x)$ and $B(x)$ are called diffusion and drift coefficients, such that $B(x) > 0$. Diffusion and drift coefficients in Eq.(1) can be functions of x and t as well as:

$$\frac{\partial u}{\partial t} = [-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)]u. \tag{2}$$

Eq.(1) is also well known as a forward Kolmogorov equation. There also exists another type of this equation which is called a backward one as [15]:

$$\frac{\partial u}{\partial t} = -[A(x,t)\frac{\partial}{\partial x} + B(x,t)\frac{\partial^2}{\partial x^2}]u. \tag{3}$$

A generalization of Eq.(1) to N variables of, $x_1, x_2, \ldots, x_N$, yields to:

$$\frac{\partial u}{\partial t} = [-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(\mathbf{x}) + \sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(\mathbf{x})]u, \tag{4}$$

with the following initial condition:

$$u(\mathbf{x},0) = f(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2, \ldots, x_N) \in R^N.$$

The nonlinear Fokker-Planck equation is a more general form of linear one which has also been applied in vast areas such as plasma physics, surface physics, population dynamics, biophysics, engineering, neurosciences, polymer physics, laser physics, nonlinear hydrodynamics, pattern formation and marketing [16]. The nonlinear form of Fokker-Planck equation can be expressed in the following form:

$$\frac{\partial u}{\partial t} = [-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)]u. \tag{5}$$

A generalization of Eq.(5) with N variables of $x_1, x_2, \ldots, x_N$, leads to:

$$\frac{\partial u}{\partial t} = [-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(\mathbf{x},t,u) + \sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(\mathbf{x},t,u)]u, \ \mathbf{x} = (x_1, x_2, \ldots, x_N) \in R^N. \tag{6}$$

3 Homotopy perturbation method (HPM)

To illustrate the basic ideas of this method, we consider the following functional equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{7}$$

with the following boundary conditions:

$$B(u, \frac{\partial u}{\partial n}) = 0, \qquad r \in \Gamma, \tag{8}$$

where A is a functional operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω.
The operator A can be decomposed into a linear part and a nonlinear one, designated as L and N respectively. Hence Eq.(7) can be written as the following form:

$$L(u) + N(u) - f(r) = 0.$$

Using homotopy technique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \longrightarrow R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0] + p[A(v) - f(r)] = 0, \tag{9}$$

where $p \in [0,1]$ is an embedding parameter u_0 and is an initial approximation for the solution of Eq.(7) which satisfies the boundary conditions. Obviously, from Eq.(9) we will have:

$$H(v,0) = L(v) - L(u_0) = 0,$$

$$H(v,1) = A(v) - f(r) = 0.$$

By changing the values of $p$from zero to unity,$v(r,p)$ change from $u_0(r)$ to $u(r)$, in topology this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. Due to the fact that can be considered as a small parameter, hence we consider the solution of Eq.(9) as a power series in p as the following:

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (10)$$

setting $p = 1$ results in an approximation solution of Eq.(7)

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \dots$$

4 Examples

To illustrate capability, reliability and simplicity of the method, six examples for different cases of the equation will be discussed here.

Example 1. Consider Fokker-Planck Eq.(1) in the case that:

$$u(x,0) = x, \quad x \in R, \qquad A(x) = -1, \qquad B(x) = 1. \quad (11)$$

Using HPM, We construct the following homotopy:

$$H(v,p) = (1-p)[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}] + p[\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2}] = 0. \quad (12)$$

Substituting Eq.(10) into Eq.(12) and equating the terms with identical powers of p, we derive:

$$\begin{aligned}
p^0 &: \frac{\partial v_0}{\partial t} - \frac{u_0}{\partial t} = 0, \qquad v_0(x,0) = x,\\
p^1 &: \frac{\partial v_1}{\partial t} + \frac{u_0}{\partial t} - \frac{\partial v_0}{\partial x} - \frac{\partial^2 v_0}{\partial x^2} = 0, \qquad v_1(x,0) = 0, \quad (13)\\
p^2 &: \frac{\partial v_2}{\partial t} - \frac{\partial v_1}{\partial x} - \frac{\partial^2 v_1}{\partial x^2} = 0, \qquad v_2(x,0) = 0,\\
&\vdots
\end{aligned}$$

Consider $u_0(x,t) = x$ as an initial approximation which satisfies the initial condition. From Eqs.(13), the following terms can be computed successively:

$$v_0(x,t) = x,$$

$$v_1(x,t) = t,$$

$$v_n(x,t) = 0, \qquad n = 2, 3, \dots$$

Therefore, the solution of Eq.(11) when $p \longrightarrow 1$ will be as follows:

$$u(x,t) = x + t.$$

which is an exact solution and is the same as that reported in [14].

Example 2. Consider Eq.(2) such that:

$$u(x,0) = sinh(x), \qquad x \in R,$$

$$A(x,t) = e^t(coth(x)cosh(x) + sinh(x)) - coth(x), \tag{14}$$

$$B(x,t) = e^t cosh(x).$$

Similarly, a homotopy can be constructed in the following form:

$$H(v,p) = (1-p)[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}] + p[(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}A(x,t) - \frac{\partial^2}{\partial x^2}B(x,t))v] = 0. \tag{15}$$

Putting Eq.(10) into Eq.(15) and collecting terms with powers of p as $0, 1, 2, \dots$, gives:

$$\begin{aligned}
p^0 &: \frac{\partial v_0}{\partial t} - \frac{u_0}{\partial t} = 0, \qquad v_0(x,0) = sinh(x),\\
p^1 &: \frac{\partial v_1}{\partial t} + \frac{u_0}{\partial t} + \frac{\partial v_0 A(x,t)}{\partial x} - \frac{\partial^2 v_0 B(x,t)}{\partial x^2} = 0, \qquad v_1(x,0) = 0, \quad (16)\\
p^2 &: \frac{\partial v_2}{\partial t} + \frac{\partial v_1 A(x,t)}{\partial x} - \frac{\partial^2 v_1 B(x,t)}{\partial x^2} = 0, \qquad v_2(x,0) = 0,\\
&\vdots
\end{aligned}$$

Let's select $u_0(x,t) = sinh(x)$ as a first approximation for the solution which satisfies the initial condition, from Eqs.(16) we derive:

$$v_0(x,t) = sinh(x),$$

$$v_1(x,t) = tsinh(x),$$

$$v_2(x,t) = \frac{t^2}{2!}sinh(x),$$

$$\vdots$$

Therefore, the solution of Eq.(14) when $p \longrightarrow 1$ will be as follows:

$$u(x,t) = sinh(x) + tsinh(x) + \frac{t^2}{2!}sinh(x) + \dots = e^t sinh(x),$$

which is an exact solution and is the same as that obtained by ADM [14].

Example 3. Consider the backward Kolmogorov Eq.(3) such that:

$$u(x,0) = x+1, \quad x \in R, \qquad A(x,t) = -(x+1), \qquad B(x,t) = x^2 e^t. \tag{17}$$

Using HPM, we construct a homotopy in the form:

$$H(v,p) = (1-p)[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}] + p[(\frac{\partial}{\partial t} + A(x,t)\frac{\partial}{\partial x} + B(x,t)\frac{\partial^2}{\partial x^2})v] = 0. \tag{18}$$

Substituting Eq.(10) into Eq.(18) and rearranging based on powers of p-terms, we obtain:

$$\begin{aligned}
p^0 &: \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \qquad v_0(x,0) = x+1,\\
p^1 &: \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} + A(x,t)\frac{\partial v_0}{\partial x} + B(x,t)\frac{\partial^2 v_0}{\partial x^2} = 0, \qquad v_1(x,0) = 0,\\
p^2 &: \frac{\partial v_2}{\partial t} + A(x,t)\frac{\partial v_1}{\partial x} + B(x,t)\frac{\partial^2 v_1}{\partial x^2} = 0, \qquad v_2(x,0) = 0,\\
&\vdots
\end{aligned} \tag{19}$$

Consider $u_0(x,t) = x+1$ as an initial approximation which satisfies the initial condition. Successive solving of Eqs.(19) leads to:

$$\begin{aligned}
v_0(x,t) &= x+1,\\
v_1(x,t) &= t(x+1),\\
v_2(x,t) &= \frac{t^2}{2}(x+1),\\
&\vdots
\end{aligned}$$

Hence, the solution of Eq.(17) when $p \longrightarrow 1$ will be in the following form:

$$u(x,t) = (x+1) + t(x+1) + \frac{t^2}{2!}(x+1) + \ldots = e^t(x+1),$$

which is an exact solution and is the same as that reported in [14].

Example 4. Consider the generalized linear Eq.(4) such that:

$$u(\mathbf{x},0) = x_1, \qquad \mathbf{x} = (x_1,x_2)^t \in R^2,$$

$$A_1(x_1,x_2) = x_1,$$

$$A_2(x_1, x_2) = 5x_2,$$

$$B_{1,1}(x_1, x_2) = x_1^2, \tag{20}$$

$$B_{1,2}(x_1, x_2) = 1,$$

$$B_{2,1}(x_1, x_2) = 1,$$

$$B_{2,2}(x_1, x_2) = x_2^2,$$

Applying HPM, we construct a homotopy as follows:

$$H(v, p) = (1 - p)[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}] + p[(\frac{\partial}{\partial t}$$

$$+ \sum_{i=1}^{2} \frac{\partial}{\partial x_i} A_i(x_1, x_2) - \sum_{i,j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}(x_1, x_2))v] = 0. \tag{21}$$

Substituting Eq.(10) into Eq.(21) and collecting terms with powers of p as 0,1,2, ... , we can obtain:

$$\begin{aligned}
p^0 &: \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \ v_0(\mathbf{x}, 0) = x_1, \\
p^1 &: \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} + \sum_{i=1}^{2} \frac{\partial v_0 A_i(x_1, x_2)}{\partial x_i} - \sum_{i,j=1}^{2} \frac{\partial^2 v_0 B_{i,j}(x_1, x_2)}{\partial x_i \partial x_j} = 0, \ v_1(\mathbf{x}, 0) = 0, \quad (22) \\
p^2 &: \frac{\partial v_2}{\partial t} + \sum_{i=1}^{2} \frac{\partial v_1 A_i(x_1, x_2)}{\partial x_i} - \sum_{i,j=1}^{2} \frac{\partial^2 v_1 B_{i,j}(x_1, x_2)}{\partial x_i \partial x_j} = 0, \ v_2(\mathbf{x}, 0) = 0, \\
&\vdots
\end{aligned}$$

Let's select $u_0(x_1, x_2, t) = x_1$ as a first approximation for the solution which satisfies the initial condition. From Eqs.(22), the following terms can be computed successively:

$$v_0(x_1, x_2, t) = x_1$$

$$v_1(x_1, x_2, t) = tx_1$$

$$v_2(x_1, x_2, t) = \frac{t^2}{2!} x_1$$

$$\vdots$$

Thus, the solution of Eq.(20) when $p \longrightarrow 1$ will be as:

$$u(x_1, x_2, t) = x_1 + tx_1 + \frac{t^2}{2!} x_1 + \ldots = e^t x_1,$$

which is an exact solution and is the same as that obtained by ADM [14].

Example 5. Consider the nonlinear Eq.(5) in the case that:

$$u(x,0)=x^2, \qquad x \in R,$$

$$A(x,t,u)=\frac{4}{x}u-\frac{x}{3}, \tag{23}$$

$$B(x,t,u)=u,$$

Using HPM, we construct the following homotopy:

$$H(v,p)=(1-p)[\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}]+p[(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}A(x,t,v)-\frac{\partial^2}{\partial x^2}B(x,t,v))v]=0. \tag{24}$$

Substituting Eq.(10) into Eq.(24) and rearranging based on powers of p-terms, we derive:

$$\begin{aligned}
p^0 &: \frac{\partial v_0}{\partial t}-\frac{u_0}{\partial t}=0, \qquad v_0(x,0)=x^2,\\
p^1 &: \frac{\partial v_1}{\partial t}+\frac{u_0}{\partial t}+\frac{\partial(\frac{4}{x}(v_0)^2-\frac{x}{3}v_0)}{\partial x}-\frac{\partial^2(v_0)^2}{\partial x^2}=0, \qquad v_1(x,0)=0,\\
p^2 &: \frac{\partial v_2}{\partial t}+\frac{\partial(\frac{8}{x}v_0v_1-\frac{x}{3}v_1)}{\partial x}-2\frac{\partial^2(v_0v_1)}{\partial x^2}=0, \qquad v_2(x,0)=0,\\
&\vdots
\end{aligned} \tag{25}$$

We select $u_0(x,t)=x^2$ as an initial approximation that satisfies the initial condition, from Eqs.(25) we will obtain:

$$\begin{aligned}
v_0(x,t)&=x^2\\
v_1(x,t)&=tx^2\\
v_2(x,t)&=\frac{t^2}{2!}x^2\\
&\vdots
\end{aligned}$$

Thus, the solution of Eq.(23) when $p \longrightarrow 1$ will be as follows:

$$u(x,t)=x^2+tx^2+\frac{t^2}{2!}x^2+\ldots=e^tx^2,$$

which is an exact solution and is the same as that reported in [14].

5 Conclusions

In this article, homotopy perturbation method has been used to obtain the exact solution of Fokker-Planck equation. It was shown that this method is very efficient and powerful to get the exact solution. Moreover, a comparison between HPM and ADM shows that although the results of the methods are the same, HPM can overcome difficulties arising in the calculation of Adomian'spolynomials. Therefore, HPM is much easier and more convenient to apply than ADM. The computations associated with examples in this article were performed using Maple 10.

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