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Bochner-Recurrent Nearly Kaehlerian Manifolds

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Abstract

In this paper it has been proved that the Bochner-recurrent nearly Kaehlerian manifold is either Bochner-symmetrical or a Bochner-recurrent Kaehlerian manifold. It has also been proved that the non-trivially Bochner-recurrent nearly Kaehlerian manifold with the recurrent Ricci tensor is either Bochner-symmetrical, or locally holomorphic isometrically to the manifold of the type $M^2 \times C^{n-1}$ with the standard Kaeherian structure.

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1 Introduction

Let M be a Riemannian manifold, T be a non-zero tensor field (tensor) of the type (r, s) on M. It is said that tensor T, is recurrent if there is 1-form ρ on M such that

$$\nabla T = \rho \otimes T,$$

where ∇ is the Riemannian connection on M. The 1-form ρ is called the recurrence covector. The Riemannian manifold which is allowing field of the recurrent tensor T is called T-recurrent. If $\rho = 0$ then the manifold is called T-symmetrical, and if $\rho \neq 0$ then it is called nontrivially T-recurrent. The Riemannian manifolds with recurrent curvature tensor of one or another form are widely studied in the differential geometry. This is mostly explained by their important role in the general theory of a relativity. The interest to these manifolds arise in connection with will-known Hadamard's problem, whether it is possible to develop the theory of gravitation and electrostatics in accordance with Newton's and Coulomb's laws within the framework of the Riemannian geometry?

The solution of this problem has lead to study of harmonic manifolds. Some example of this manifolds were obtained by Ruse, they are found to be the manifolds with recurrent tensor R of Riemannian curvature [2]. This work by Ruse was the starting point of the study of the Riemannian manifolds with the recurrent tensor of Riemannian curvature, moreconcisely the recurrent of Riemannian manifolds. The existence of the recurrent tensor of the type $(3,1)$ on Riemannian manifold lay on a powerful restriction on the geometry of this manifold. For example, it is known that a complete Riemannian manifold with the recurrent curvature tensor is a factor manifold of 1 -connected of Riemannian symmetrical space or manifold of the type $M^2 \times R^{n-2}$ according to the discrete subgroup of isometrics [3].

Some authors, studied the Kaehlerian manifolds with the recurrent curvature tensor of one or another form [4, 5].

In the present work, we give the profound description of an important class of nearly Kaehlerian manifolds with the recurrent Bochner curvature tensor (Bochner-recurrent NK-manifolds), by generalization the well-known results in this direction.

2 Preliminaries

Let M be an almost Hermitian manifold with an almost Hermition structure $(J, g = \langle ., . \rangle)$, $J^2 = -id$, $\langle JX, JY \rangle = \langle X, Y \rangle$, $\dim(M) = 2n > 2$, and let $\chi(M)$ be the Lie algebra of C^∞ vector fields on M, ∇ be the Riemannian connection and d is the operator of the exterior differentiation.

It is well known that representation of an almost Harmitian structure on M is equivalent to the representation of G-structure on M with the structural group $U(n)$, this group is a space which elements are the frames consisting of pairwise of dual eigenvectors of structural endomorphism J and unitary in the natural Hermitian metric of complexification of the corresponding tangent space (such frames are called A-frame) [7]. This G-structure is called adjoined. We will presuppose that indices $i, j, k, ...$ range from 1 to $2n$, the indices a, b, c, d, f, g, h range from 1 to n and $\hat{a} = a + n$, . We will denote by $[a, ..., b]$ and $(a, ..., b)$ the alternation and symmetrization by the indices $a, ..., b$ respectively. It is easy verify that in the associated space of the G-structure the components of tensors g and J are given by the following matrices

$$(g_{ij}) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}; \left(J^i_j\right) \rightleftarrows \begin{pmatrix} iI_n & 0 \\ 0 & -iI_n \end{pmatrix},$$

where I_n- is the identity matrix of order n.

From[6, 7], we note that an almost Hermitian structure is called nearly Kaehlerian (in short, NK-) structure if its fundamental form $\Omega(X, Y) = (X, JY)$ is a killing form or that is equivalent to

$$\nabla_X (J) X = 0;\ X \in \chi(M).$$

The nearly Kaehlerian structure is called Kaehlerian if $\nabla J = 0$ otherwise it's proper. It is well known [7] that the structural equations of the Riemannian connection of a nearly Kaehlerian structure on the space of a joined G-structure (called structural equations of the nearly Kaehlerian structure) have the following form:

$$\begin{array}{rcl} 1.\ d\omega^a & = & \omega^a_b \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c; \\ 2.\ d\omega_a & = & -\omega^b_a \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c; \\ 3.\ d\omega^a_b & = & \omega^a_c \wedge \omega^c_b + \left(2B^{adh}B_{hbc} + A^{ad}_{bc}\right)\omega^c \wedge \omega_d; \end{array} \tag{1}$$

where $\omega_a = \omega^{\hat{a}}$, $\left\{B^{abc}\right\}$, $\{B_{abc}\}$ and $\left\{A^{ad}_{bc}\right\}$ are the system of functions in the associated space of the G-structure serving the components of the complex tensors on M. these tensors are called Kirichenko structural tensors of the first, second and third order respectively. Being so, the structural tensors of the first and second order are skew-symmetric and the structural tensor of the third order is symmetrical by any pair of superscript or subscript [7].

In addition it is known that the components of the Kirichenko structural tensors of the first and second order of a NK-manifold are satisfy the following identities

$$\begin{array}{rcl} 1.\ dB^{abc} - B^{hbc}\omega^a_h - B^{ahc}\omega^b_h - B^{abh}\omega^h_c & = & 0; \\ 2.\ dB_{abc} - B_{hbc}\omega^h_a - B_{ahc}\omega^h_b - B_{abh}\omega^c_h & = & 0; \end{array} \tag{2}$$

The relationship(2) is equivalent to parallelism of these tensor in the associated connection [8] and components of the Kirichenko structural tensor of the third order are satisfy the following identity:

$$dA^{ad}_{bc} + A^{ad}_{hc}\omega^h_b + A^{ad}_{bh}\omega^h_c - A^{hd}_{bc}\omega^a_h - A^{ah}_{bc}\omega^d_h = A^{ad}_{bch}\omega^h + A^{adh}_{bc}\omega_h; \tag{3}$$

where $\left\{A^{ad}_{bch}\right\}$ and $\left\{A^{adh}_{bc}\right\}$ are the system of functions in the space of the a joined G-structure that are used for the components of the covariant differential structural tensor of the third order in the associated connection, and they are symmetrical by any pair of superscript or subscript, (we call to mind that the associated connection (extended) of almost Hermitian manifold is called the connection $\tilde{\nabla} = \nabla + T$, where T is the composite tensor of the adjoint Q-algebra [9]). As well we call to mind that the components of structural tensor of NK-manifold are satisfying the formula of complex continuation [7]:

$$\bar{B}^{abc} = B_{abc};\ \bar{A}^{ad}_{bc} = A^{bc}_{ad}; \tag{4}$$

We can note that the structural tensor of the 1st and 2nd order vanish precisely, when the manifold is Kaehlerian [7]. For convenience we will take the following notations:

$$B^{ad}_{bc} = B^{adh}B_{hbc};\ B^{a}_{b} = B^{ac}_{bc};\ B = B^{a}_{a};\ A^{a}_{b} = A^{ac}_{bc}, A = A^{a}_{a}.$$

Let R, Q, $\aleph$ are Riemann-Christofel tensor, Ricci tensor and scalar curvature of the manifold M respectively. We call to mind that the main nonzero components of these tensors on the space of the G-structure (i.e. in an A-frame) for a NK-manifold have the following form respectively[10]:

$$\begin{aligned} R_{\breve{a}\hat{b}cd} &= -2B^{ab}_{cd};\ R_{\hat{a}bc\breve{d}} = B^{ad}_{bc} + A^{ad}_{bc}; R_{abcd} = R_{\hat{a}bcd} = 0; \\ Q_{ab} &= 0;\ Q_{\hat{a}b} = 3B^{a}_{b} - A^{a}_{b};\ \aleph = 6B - 2A. \end{aligned} \tag{5}$$

The other nonzero components of these tensors are calculated with regard to the reality and classical characteristics of symmetry of these tensors.

Definition 1 [11]. The Bochner tensor of an almost Hermitian manifold M^{2n} is called the tensor $\tilde{B}$ of the type $(4,0)$ on M^{2n} which is defined by the formula:

$$\begin{aligned} \tilde{B}(X,Y,Z,W) &= R(X,Y,Z,W) + L(X,W)g(Y,Z) - L(X,Z)g(Y,W) \\ &+ L(Y,Z)g(X,W) - L(Y,W)g(X,Z) + L(JX,W)g(JY,Z) \\ &- L(JX,Z)g(JY,W) + L(JY,Z)g(JX,W) - L(JY,W)g(JX,Z) \\ &- 2L(JX,Y)g(JZ,W) - 2L(JZ,W)g(JX,Y), \end{aligned}$$

where $L(X,Y) = -\frac{1}{2n+4}Q(X,Y) + \frac{\aleph}{2(2n+2)(2n+4)}g(X,Y)$ the Ricci-Bochner bilinear form, $X,Y,Z,W \in \chi(M^{2n})$;

It will known [10] that this tensor having all symmetry properties of the Riemann-Christofel tensor R.(the tensor of Riemannian curvature).

As is shown in [1] the main nonzero component of the Bochner tensor in the space of the G-structure are the following:

$$\begin{aligned} 1.\tilde{B}_{abcd} &= \tilde{B}_{\hat{a}bcd} = 0; \\ 2.\tilde{B}_{\hat{a}\hat{b}cd} &= -2B^{ab}_{cd}; \\ 3.\tilde{B}_{\hat{a}bc\hat{d}} &= B^{ad}_{bc} + A^{ad}_{bc} + \frac{4}{n+2}\left(3B^{(a}_{(b}\delta^{d)}_{c)} - A^{(a}_{(b}\delta^{d)}_{c)}\right) - 4\tilde{\aleph}\tilde{\delta}^{ad}_{bc} \end{aligned} \tag{6}$$

where $\tilde{\delta}^{ad}_{bc} = \delta^{a}_{b}\delta^{d}_{c} + \delta^{a}_{c}\delta^{d}_{b}$, $\tilde{\aleph} = \frac{\aleph}{2(2n+2)(2n+4)}$ and the other nonzero components of this tensor are calculated with regard to the reality and classical characteristics of symmetry of the Bochner tensor.

3 Main Results

Now suppose that M^{2n} is a B-recurrent of NK-manifold, then $\nabla \tilde{B} = \rho \otimes \tilde{B}$ which has the following writing coordinate form.

$$\tilde{B}^i_{jkl,t} = \rho_t \tilde{B}^i_{jkl};$$

we consider this identity in the fiber space of A-frames i.e. $\tilde{B}_{\hat{a}bc\hat{d},h} = \rho_h \tilde{B}_{\hat{a}bc\hat{d}}$, where comma (as usuall) is an index symbol of the covariant differentiation in Riemannian connection. By virtue of (6) part (3) and $Q^a_b = 3B^a_b - A^a_b$, we have

$$\begin{aligned} & A^{ad}_{bch} + \frac{1}{n+2}\left(Q^a_{bh}\delta^d_c + Q^d_{ch}\delta^a_b + Q^a_{ch}\delta^d_b + Q^d_{bh}\delta^a_c\right) - 4\tilde{\aleph}_h \tilde{\delta}^{ad}_{bc} \\ = \ & \rho_h \left[A^{ad}_{bc} + B^{ad}_{bc} + \frac{1}{n+2}\left(Q^a_b\delta^d_c + Q^d_c\delta^a_b + Q^a_c\delta^d_b + Q^d_b\delta^a_c\right) - 4\tilde{\aleph}\tilde{\delta}^{ad}_{bc}\right]; \end{aligned} \quad (7)$$

It is known that [7] $Q^a_b = 3B^a_b - A^a_b$, then $Q^a_{bh} = -A^a_{bh}$; Substituting in (7), we have

$$\begin{aligned} & A^{ad}_{bch} - \frac{1}{n+2}\left(A^a_{bh}\delta^d_c + A^d_{ch}\delta^a_b + A^a_{ch}\delta^a_b + A^d_{bh}\delta^a_c\right) - 4\tilde{\aleph}_h \tilde{\delta}^{ad}_{bc} \\ = \ & \rho_h \left[A^{ad}_{bc} + B^{ad}_{bc} + \frac{1}{n+2}\left(Q^a_b\delta^d_c + Q^d_c\delta^a_b + Q^a_c\delta^d_b + Q^d_b\delta^a_c\right) - 4\tilde{\aleph}\tilde{\delta}^{ad}_{bc}\right] \end{aligned} \quad (8)$$

by contracting (8) by indices **a** and **b**, we have

$$\begin{aligned} & A^d_{ch} - \frac{1}{n+2}\left(A_h\delta^d_c + nA^d_{ch} + A^d_{ch} + A^d_{ch}\right) - 4\tilde{\aleph}_h (n+1)\,\delta^d_c \\ = \ & \rho_h \left[A^d_c + B^d_c + \frac{1}{n+2}\left(Q\delta^d_c + nQ^d_c + Q^d_c + Q^d_c\right) - 4\tilde{\aleph}(n+1)\,\delta^d_c\right]; \end{aligned}$$

$$\begin{aligned} & A^d_{ch} - \frac{1}{n+2}(n+2)\,A^d_{ch} - \frac{1}{n+2}A_h\delta^d_c - 4\tilde{\aleph}_h (n+1)\,\delta^d_c \\ = \ & \rho_h \left[A^d_c + B^d_c + \frac{1}{n+2}(n+2)Q^d_c + \frac{1}{n+2}Q\delta^d_c - 4\tilde{\aleph}(n+1)\,\delta^d_c\right]; \end{aligned}$$

$$\begin{aligned} & -\frac{1}{n+2}A_h\delta^d_c - 4\tilde{\aleph}_h (n+1)\,\delta^d_c \\ = \ & \rho_h \left[A^d_c + B^d_c + Q^d_c + \frac{1}{n+2}Q\delta^d_c - 4\tilde{\aleph}(n+1)\,\delta^d_c\right] \end{aligned} \quad (9)$$

but $\aleph = Q_i^i = 2Q_a^a = 6B_a^a - 2A_a^a$, consequently $\aleph_h = -2A_a^a$, then

$$\tilde{\aleph}_h = \frac{-2A_h}{8(n+1)(n+2)} = \frac{-A_h}{4(n+1)(n+2)};$$

Substituting in (9) $\tilde{\aleph}_h$ and with $Q_c^d = 3B_c^d - A_c^d$, we can see that

$$-\frac{1}{n+2}A_h\delta_c^d + \frac{4(n+1)A_h}{4(n+1)(n+2)}\delta_c^d = \rho_h\left[\left(4B_c^d + \frac{1}{n+2}Q\delta_c^d\right) - 4(n+1)\tilde{\aleph}\delta_c^d\right];$$

Hence,

$$\begin{aligned} 0 &= \rho_h\left[\left(4B_c^d + \frac{1}{n+2}Q\delta_c^d\right) - 4(n+1)\cdot\frac{6B-2A}{8(n+1)(n+2)}\delta_c^d\right] \\ 0 &= \rho_h\left[\left(4B_c^d + \frac{1}{n+2}Q\delta_c^d\right) - \frac{3B-A}{n+2}\delta_c^d\right] \\ &= \rho_h\left[\left(4B_c^d + \frac{1}{n+2}Q\delta_c^d\right) - \frac{Q}{n+2}\delta_c^d\right] = 4\rho_h B_c^d = 0; \end{aligned}$$

Consequently either

1. $\rho_h = 0$; i.e. $\nabla B = 0 \Rightarrow M^{2n}$-Bochner symmetrical manifold.

or,

2. $B_c^d = 0$; i.e $B^{dah}B_{cah} = 0$, by contracting by indices **d** and **c** we have $B^{cah}B_{cah} = 0 \Rightarrow \sum_{c,a,h=1} \bar{B}_{cah}B_{cah} = 0 \Rightarrow \sum_{c,a,h=1} |B_{cah}|^2 = 0 \Rightarrow B_{cah} = 0$, i.e M^{2n} is a Kaehlerian manifold [7].

Thus we have proved the following result:

Theorem 1. The Bochner recurrent (B-recurrent) NK-manifold is a Bochner symmetrical manifold or a Bochner recurrent Kaehlerian manifold.

Now assume that M^{2n} is a B-recurrent Kaehlerian manifold, then we can rewrite formula (6) part (3) in the following form

$$\tilde{B}_{\hat{a}bc\hat{d}} = A_{bc}^{ad} - \frac{4}{n+2}A_{(b}^{(a}\delta_{c)}^{d)} - 4\tilde{\aleph}\tilde{\delta}_{bc}^{ad};$$

Let us differentiate this relationship with regard to B-recurrence of the manifold

$$A_{bch}^{ad} - \frac{4}{n+2}A_{h(b}^{(a}\delta_{c)}^{d)} - 4\tilde{\aleph}_h\tilde{\delta}_{bc}^{ad} = \rho_h\left[A_{bc}^{ad} - \frac{4}{n+2}A_{(b}^{(a}\delta_{c)}^{d)} - 4\tilde{\aleph}\tilde{\delta}_{bc}^{ad}\right]$$

where $A_{bc}^{a} = A_{hbc}^{ha}$.

On the other hand, by breaking down the relationship

$$dQ_{ij} + Q_{kj}\omega_i^k + Q_{ik}\omega_j^k = Q_{ij,k}\omega^k$$

(which is equivalent to that Q is a tensor of the type $(2,0)$) with regard to (5) we have $A^a_{bc} = -Q^a_{b,c}$, and hence

$$A^{ad}_{bch} + \frac{4}{n+2} Q^{(a}_{h,(b}\delta^{d)}_{c)} - 4\tilde{\aleph}_h \tilde{\delta}^{ad}_{bc} = \rho_h \left[A^{ad}_{bc} + \frac{4}{n+2} A^{(a}_{(b}\delta^{d)}_{c)} - 4\tilde{\aleph}\tilde{\delta}^{ad}_{bc} \right]; \qquad (10)$$

In particular, assume that M^{2n} is $BQ-$recurrent, i.e. it has additionally the recurrent Ricci tensor (with the same recurrence covector) then obvious that $Q^a_{h,b} = Q^a_{b,h} = \rho_h Q^a_b, \tilde{\aleph}_h = \rho_h \tilde{\aleph}$, and by virtue of(10) taking the form $A^{ad}_{bch} = \rho_h A^{ad}_{bc}$,means tnat lt is a recurrent [7]; with the account of [7], M^{2n} is locally holomorphic isometrical to the product of the two-dimensional Kaehlerian manifold M^2 by the complex Euclidean space C^{n-1} with the standard Kaehlerian structure. Taking into account Theorem 1, we get:

Theorem 2. The $BQ-$recurrent nearly Kaehlerian manifold is either B-symmetric or locally holomorphic isometrically to the manifold of the type $M^2 \times C^{n-1}$ with the standard Kaehlerian structure.

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