

# Scattering Matrix and Spectral Shift Function with Potentials in Kato Class

Mahmoud-Sabry Saif and Usama M. Abdelsalam<sup>1</sup>

Mathematics Department, Faculty of Science  
Fayoum University, Egypt

## Abstract

The scattering matrix and the spectral shift function are studied for a model of perturbation theory where the Hamiltonian  $H$  is a sum of a multiplication operator  $H_0$  by  $|x|^{2l}$ ,  $l > 0$  in the space  $L_2(R^d)$  and of  $V$ , such that  $V \in K_\nu$  (Kato class).

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## 1 Introduction

Our aim here is to study the scattering problem for a pair  $H_0, H$  where  $H_0$  is the multiplication operator by  $|x|^{2l}$ ,  $l > 0$  in the space  $L_2(R^d)$  and  $H = H_0 + V$ ,  $V \in K_\nu$  (Kato class), in particular the spectral shift function and the spectrum of the scattering matrix are studied.

As an example we can consider  $H_0 = (-\Delta)^l$ ,  $l > 0$ , a fraction laplacian operator, and  $V = V_+ - V_-$ ,  $V_- \in K_\nu$ ,  $V_+ \in K_\nu^{loc}$ , such that  $V \in l^1(L_2(R^\nu))$ .

The starting point of scattering theory is that the absolutely continuous part of self-adjoint operator is stable under fairly general perturbations.

We use the trace class method[see 1,3,6,8 ] to prove the existence and completeness of  $W_\pm(H, H_0)$ . The fundamental theorem for the trace class method is the Kato-Rosenblum theorem[6,8] which state that if the difference  $V = H - H_0$

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<sup>1</sup>maths\_us@hotmail.com

belongs to the trace class, then the wave operators  $W_{\pm}(H, H_0)$  exist and are complete.

If  $V$  is in some sense compact relative to  $H_0$ , then the essential spectra of the operators  $H_0$  and  $H = H_0 + V$  coincide[3].

Scattering operator is constructed in a known way from wave operators,  $S = W_{+}^{*}W_{-}$ , and give rise in turn to  $SM$  or scattering matrix  $s(\lambda) = s(H, H_0; \lambda)$ ,  $\lambda > 0$ , we study the scattering matrix, in particular we find the essential spectrum of  $SM$ . The spectrum of the scattering matrix was for the first time considered by Birman and Krein in the paper [2]. In this paper perturbations of trace class type were studied.

The mathematical theory of the spectral shift function was constructed by Krein in [5], one of his results can be formulated in the following way. Let  $H_0$  and  $H$  be self adjoint operators with trace class difference  $V = H - H_0$ , then there exists a function  $\xi(\lambda) = \xi(\lambda; H, H_0)$ ,  $\xi \in L_1$ , known as the spectral shift function, such that the trace formula

$$Tr(f(H) - f(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda)f'(\lambda)d\lambda, \quad \xi(\lambda) = \xi(\lambda; H, H_0),$$

holds at least for all functions  $f \in C_0^{\infty}(R)$ .

In thier paper [2] Birman and Krein found a connection between scattering theory and the theory of the spectral shift function. Actually, they showed that

$$\det s(H, H_0; \lambda) = e^{-2\pi i\xi(\lambda; H, H_0)}$$

for almost all  $\lambda$  from the core of the spectrum of the operator  $H_0$ . coincide.

## 2 Preliminary Notes

Here a necessary background of scattering theory is described for a pair of self-adjoint operators  $H_0, H$  in a Hilbert space  $X$ . Wave operators for the pair  $H_0, H$  are introduced as strong limits

$$W_{\pm}(H, H_0) = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} P_0^{ac}.$$

where  $P_0^{ac}$  is the orthogonal projection onto the absolutely continuous subspace  $X_0^{ac}$  of the operator  $H_0$ . If  $W_{\pm}$  exist, then they are isometric on  $X_0^{ac}$  and have the intertwining property  $HW_{\pm} = W_{\pm}H_0$ . The scattering operator  $S = W_{+}^{*}W_{-}$

commute with  $H_0$ . If the ranges  $RanW_{\pm}$  of  $W_{\pm}$  coincide with the absolutely continuous subspace  $X^{ac}$  of the operator  $H$ , then wave operators are called complete. In this case  $S$  is a unitary operator in  $X_0^{ac}$ .

Consider now the diagonalization of  $H_0$  under the representation of  $X_0^{ac}$  as a direct integral

$$X_0^{ac} \leftrightarrow \int_{\sigma_0} \oplus x(\lambda) d\lambda. \tag{2.1}$$

Here  $\sigma_0$  denotes the core of the spectrum of  $H_0$ , i.e. it is some set of minimal Lebesgue measure which carries the spectral measure  $E_0(\cdot)$  of  $H_0$ . The direct integral in the RHS of (2.1) is the  $L_2$ -space of vector-functions defined on  $\sigma_0$  and taking values in auxiliary Hilbert spaces  $x(\lambda)$ . By a unitary operator  $U$  the operator  $USU^*$  acts as multiplication by an operator-function  $s(\lambda) : x(\lambda) \rightarrow x(\lambda)$  which is unitary for a.a.  $\lambda \in \sigma_0$  and called the scattering matrix (SM).

**Lemma 2.1** [8] If  $H \in B_r, r > 0$  and  $A \in B$ , then  $AH \in B_r$  and  $HA \in B_r$ , where  $B, B_1, B_2, B_{\infty}$  are, respectively, the sets of all bounded, trace class, Hilbert-Schmidt and compact operators on  $X$ .

Kato and Rosenblum [8] proved the existence of the wave operators for a pair of self-adjoint operators with a trace class difference.

**Theorem 2.1** Let  $H_0$  and  $H$  be self-adjoint operators in spaces  $X_0$  and  $X$ , respectively,  $J : X_0 \rightarrow X$  is a bounded operator and  $V = HJ - JH_0 \in B_1$ . Then the wave operators  $W_{\pm}(H, H_0, J)$  exist and are complete.

Kuroda and Birman extend this theorem by the idea of using the resolvent [8] where in some applications we cannot use the Kato-Rosenblum theorem as  $V$  isn't trace class.

**Kuroda-Birman theorem 2.2**

Let  $H_0$  and  $H$  be self-adjoint operators so that

$$(H + I)^{-1} - (H_0 + I)^{-1} \in B_1 \tag{2.2}$$

then WO  $W_{\pm}(H_0, H)$  exist and are complete.

The operator  $H_0 = (-\Delta)^l$  is selfadjoint where it is a function of self-adjoint operator and also the action  $H_0$  reduces to multiplication by the function  $|x|^{2l}$ . More exactly, let  $G$  denote multiplication by  $|x|^{2l}$  in  $L_2(R^d)$ . Then for an

arbitrary (bounded) function  $f$  we have

$$Ff(H_0) = f(G)F. \quad (2.3)$$

where  $F$  denotes the Fourier transformation. In particular, the resolvent  $R_0 = (H_0 - z)^{-1}$  of the operator  $H_0$  in the impulse representation acts as multiplication by  $(|x|^{2l} - z)^{-1}$ .

We consider the identity

$$(H + I)^{-1} - (H_0 + I)^{-1} = -(H + I)^{-1}V(H_0 + I)^{-1} \quad (2.4)$$

connecting the resolvents of the operators  $H_0$  and  $H$ .

Now we can check that the wave operators

$$W_{\pm}(H, H_0) = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}$$

exist and are complete, that is their ranges coincide with the absolutely continuous subspace of the operator  $H$ . According to Kuroda-Birman theorem, to that end it suffices to verify that operator

$$(H + I)^{-1} - (H_0 + I)^{-1} \in B_1$$

let us now use the resolvent identity

$$(H + I)^{-1} - (H_0 + I)^{-1} = -((H + I)^{-1}(H_0 + I))((H_0 + I)^{-1}V(H_0 + I)^{-1}$$

the first factor in the right-hand side is bounded because  $V$  is bounded and

$$(H + I)^{-1}(H_0 + I) = Id - (H + I)^{-1}V.$$

So, we have to prove that the second is in trace class i.e;

$$(H_0 + I)^{-1}V(H_0 + I)^{-1} \in B_1 \quad (2.5)$$

Spectral and perturbation theories for unitary operators in a Hilbert space  $X$  are essentially similar to those for the self-adjoint case. The spectral measure  $E_u(\Delta)$  of a unitary operator  $U$  is defined on a Borel sets  $\Delta$  of the unit circle  $\mathbb{T} : |\Delta|$  is the Lebesgue measure of  $\Delta$  (so that  $|\mathbb{T}| = 2\pi$ ). The essential spectrum  $\sigma_U^{(ess)}$  of  $U$  consists of its whole spectrum  $\sigma_U = \text{supp}E_U$  without isolated eigenvalues of finite multiplicity. We denote by  $(\nu_1, \nu_2)$  and  $[\nu_1, \nu_2]$ , where  $|\nu_i| = 1$ , the corresponding open and closed arcs of the unit circle  $\mathbb{T}$  swept out as  $\nu_1$  moves to  $\nu_2$  in the counterclockwise (which we designate as the positive) direction. Let  $\beta$  be the class of such unitary operators  $U$  that

$U - I \in B_\infty$ , where  $I$  is the identity operator.

**Theorem 2.3** The spectrum of an operator  $U \in \beta$  consists of eigenvalues accumulating only at the point 1. Eigenvalues distinct from 1 have finite multiplicity.

We can define  $V_+$  and  $V_-$  as follows:

$V_- = \max(-V, 0)$  lie in  $K_\nu$ ,  $V_+ = \max(V, 0)$  lie in  $K_\nu^{loc}$

**Theorem 2.4** Let  $V_- \in K_\nu, V_+ \in K_\nu^{loc}$ . Let  $f$  be a bounded Borel function on  $\text{spec}(H)$  obeying

$$|f(x)| \leq c(1 + |x|)^{-\alpha}$$

for some  $\alpha > \frac{\nu}{2}$ . Let  $g \in l^1(L^2(R^\nu))$ . Then  $f(H)g(x)$  is in  $B_1$ .

### 3 Main Results

Let us first discuss the existence and completeness of the wave operators  $W_\pm(H, H_0)$  where  $H_0$  is the multiplication operator by  $|x|^{2l}, l > 0$  in the space  $L_2(R^\nu)$  and  $H = H_0 + V, V = V_+ - V_-, V_- \in K_\nu, V_+ \in K_\nu^{loc}$ , such that  $V \in l^1(L_2(R^\nu))$ . We want to prove the existence and completeness of the wave operators  $W_\pm(H, H_0)$ . By Birman theorem we need to prove (2.5).

It sufficient to prove that

$$(H_0 + I)^{-1}V(H_0 + I)^{-1} \in B_1$$

Since  $(H_0 + I)^{-1}$  is a bounded operator, then by lemma 2.1 we need to show that

$$V(H_0 + I)^{-1} \in B_1. \tag{3.1}$$

since  $V \in l^1(L_2(R^\nu))$  and  $(H_0 + I)^{-1} = f(H_0)$  where  $f(x) = (1 + |x|^{2l})^{-1}$  satisfies  $f(x) \leq (1 + |x|)^{-\alpha}$  where  $\alpha = 2l$  and for  $\alpha > \frac{\nu}{2}$  hence  $2m > \frac{\nu}{2}$  tends to  $l > \frac{\nu}{4}$  which satisfies theorem 2.4 then  $V(H_0 + I)^{-1} \in B_1$  and  $(H_0 + I)^{-1}V(H_0 + I)^{-1} \in B_1$  i.e;  $(H + I)^{-1} - (H_0 + I)^{-1} \in B_1$  which complete the proof of the existence and completeness of  $W_\pm(H, H_0)$ .

To prove that the essential spectra of the operator  $H_0$  and  $H = H_0 + V$  coincide it suffices to show that  $V$  is compact relative to  $H_0$ . so by using the generalized Weyl theorem and the resolvent identity, it is sufficient to show that

$$(H_0 + I)^{-1}V(H_0 + I)^{-1} \in B_\infty.$$

**Theorem 3.1** Let  $H_0$  is the multiplication operator by  $|x|^{2l}$ ,  $l > 0$  in the space  $L_2(R^\nu)$  and  $H = H_0 + V$ ,  $V = V_+ - V_-$ ,  $V_- \in K_\nu$ ,  $V_+ \in K_\nu^{loc}$ , such that  $V \in l^1(L_2(R^\nu))$  then for  $l > \frac{\nu}{4}$ , the wave operators  $W_\pm(H, H_0)$  exist and are complete and the essential spectra of the operator  $H_0$  and  $H = H_0 + V$  coincide.

Another important object of scattering theory is the scattering operator

$$S = W_+^* W_-$$

it is unitary on  $X_0^{ac}$  if both  $W_\pm$  are complete and since it commutes with  $H_0$ , it acts as multiplication by the operator-function (scattering matrix)  $s(\lambda)$  which is unitary in  $x(\lambda)$ ,  $\lambda \in \sigma_0 \subset [0, \infty)$ . By theorem 2.3 if  $s(\lambda) - I_\lambda$  is compact, it follows that the spectrum of the operator  $s(\lambda)$  consists of eigenvalues of finite multiplicity lying on the unite circumference and accumulating at the point 1 (the point 1 may also be an eigenvalue of finite or infinite multiplicity).

By the following theorem of Birman and Krein we can check the compactness of the operator  $T(\lambda) = s(\lambda) - I_\lambda$

**Theorem 3.2 [2]**

If the operators  $H, H_0$  satisfy that  $R(z) - R_0(z) \in B_1$ , then the scattering matrix is of the form

$$s(\lambda) = I_\lambda + T(\lambda),$$

for almost all  $\lambda \in \sigma_0$ , where  $I_\lambda$  is the identity operator and  $T(\lambda)$  is an operator of trace class in  $x(\lambda)$ .

By our discussion in the previous section and theorem 2.3, we can prove that  $R(z) - R_0(z) \in B_1$  for the operators  $H, H_0$  in section 1 and we can deduce the following theorem.

**Theorem 3.3** For the pair  $H, H_0$  such that  $H$  is a sum of a multiplication operator  $H_0$  by  $|x|^{2l}$ ,  $l \geq 0$  in the space  $L_2(R^\nu)$  and  $H = H_0 + V$ ,  $V = V_+ - V_-$ ,  $V_- \in K_\nu$ ,  $V_+ \in K_\nu^{loc}$ , such that  $V \in l^1(L_2(R^\nu))$  and  $l > \frac{\nu}{4}$ . Then the spectrum of the scattering matrix  $s(H, H_0; \lambda)$  consists of eigenvalues of finite multiplicity lying on the unite circumference and accumulating at the point 1 (the point 1 may also be an eigenvalue of finite or infinite multiplicity).

The spectral shift function  $\xi(\lambda) = \xi(\lambda; H, H_0)$  is a function related to the change in the spectral density in going from  $H_0$  to  $H$ . By definition  $\xi(\lambda)$  is called a spectral shift function if for all functions  $f \in C_0^\infty(R)$  that the trace

formula

$$\text{Tr}(f(H) - f(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda) f'(\lambda) d\lambda. \tag{i}$$

holds.

Let  $\Delta(z) = \det(I + (H - H_0)R_0(z))$  and assume that the function  $f$  has two locally bounded derivatives, and

$$(\lambda^2 f'(\lambda))' = O(|\lambda|^{-1-\varepsilon}), \quad |\lambda| \rightarrow \infty, \tag{3.2}$$

such that

$$\lim_{\lambda \rightarrow -\infty} f(\lambda) = \lim_{\lambda \rightarrow +\infty} f(\lambda), \quad \lim_{\lambda \rightarrow -\infty} \lambda^2 f'(\lambda) = \lim_{\lambda \rightarrow +\infty} \lambda^2 f'(\lambda). \tag{3.3}$$

By the following theorem we can study the theory of the spectral shift function for our model, and we can find the connection between scattering theory and the theory of the spectral shift function. Actually, we can show that

$$\text{Dets}(H, H_0; \lambda) = e^{-2\pi i \xi(\lambda; H, H_0)}$$

for almost all  $\lambda$  from the core of the spectrum of the operator  $H_0$ .

**Theorem 3.4 [4]**

Let  $H$  and  $H_0$  be self-adjoint operators such that  $R(z) - R_0(z) \in B_1$  and  $f$  satisfy (3.2),(3.3). Then there exists a spectral shift function  $\xi(\lambda)$  which satisfies (i).

Also  $\xi(\lambda)$  satisfies

(ii)  $\int |\xi(\lambda)|(1 + \lambda^2)^{-1} d\lambda < \infty,$

(iii)  $\xi(\lambda) = (\frac{1}{\pi}) \text{arg}_{\varepsilon \rightarrow 0} \Delta(\lambda + i\varepsilon),$

(iv)  $\text{dets}(H, H_0; \lambda) = e^{-2\pi i \xi(\lambda; H, H_0)}$  for almost all  $\lambda$  from the core of the spectrum of the operator  $H_0$ .

From our discussion in section 1 we can deduce that the pair  $H, H_0$  can satisfy the last theorem for  $l > \frac{\nu}{4}$ .

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