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A Sequential Linear Programming Method for Generalized Linear Complementarity Problem[1]

Chuanliang Xu

Rizhao Vocational and Technical College
Rizhao Shandong, 276800, China
xchl-red@163.com, clx7007@eyou.com

Hong-Ge Zhao

Shandong Water Polytechnic College
Rizhao Shandong, 276800, China

Abstract

In this paper, we propose a sequential linear programming method for the generalized linear complementarity problem. Under suitable conditions, we show that the method terminates in a finite number of steps at a solution or stationary point of the problem. Numerical experiments of the method are reported in this paper.

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1 Introduction

The generalized linear complementarity problem, denoted by GLCP, is to find two vectors x^* and $y^* \in R^n$ such that

$$Mx^* - Ny^* \in \mathcal{K}, \quad x^* \geq 0, \quad y^* \geq 0, \quad (x^*)^\top y^* = 0,$$

where $M, N \in R^{m\times n}$ are two given matrices, and $\mathcal{K} := \{Qz + q | z \in R^l\}$ ($Q \in R^{m\times l}$ and $q \in R^m$) is an affine subspace in R^n. The GLCP is a special case of the extended linear complementarity (XLCP) which was firstly introduced by Mangasarian and Pang ([1]) and was later proved to be equivalent

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to the generalized LCP considered in [3] by Gowda ([2]). Many well-known linear complementarity problem, such as the vertical linear complementarity problem, the horizontal linear complementarity problem, the mixed linear complementarity problem, can be reformulated explicitly as The GLCP. This problem was further developed by Xiu et al. in [4]. The generalized complementarity problem plays a significant role in economics, engineering and operation research, etc. For example, the balance of supply and demand is central to all economic systems; mathematically, this fundamental equation in economics is often described by a complementarity relation between two sets of decision variables. Furthermore, the classical Walrasian law of competitive equilibria of exchange economies can be formulated as a generalized nonlinear complementarity problem in the price and excess demand variables ([5]).

In recent years, many effective methods have been proposed for solving GLCP. The basic idea of these methods is to reformulate the problem as an unconstrained or simply constrained optimization problem ([4, 6]). Different from the algorithms listed above, we propose a sequential linear programming algorithm (SLPA) to solve the GLCP in this paper based on the algorithm given by Bennett and Mangasarian in [7] for solving bilinear program problem. Under suitable conditions, we show that the method terminates in a finite number of steps at a solution or stationary point of GLCP, numerical experiments shows that the method is encouraging and promising.

Some notations used in this paper are in order. The identity matrix in real space of arbitrary dimension will be denoted by I, while a column vector of ones of arbitrary dimension will be denoted by e, "*arg vertex partial min*" denotes a vertex in the solution set of the indicated linear program. For simplicity, we use (x, y, z) for column vector $(x^\top, y^\top, z^\top)^\top$.

2 The equivalent reformulation of the GLCP

First, we give some equivalent expressions of the solution set of the GLCP.

Certainly, (x^*, y^*) is a solution of the GLCP if and only if there exist $z^* \in R^l$ such that

$$\begin{cases} x^* \geq 0, y^* \geq 0, \\ Mx^* - Ny^* - Qz^* - q = 0 \\ (x^*)^\top (y^*) = 0 \end{cases}$$

Denote $\omega^* \triangleq (x^*, y^*, z^*) \in R^{n+n+l}$. From discussion above, the GLCP can be equivalently reformulated as the following problem: find ω^* such that

$$\begin{cases} (I,0,0)\omega^* \geq 0, \quad (0,I,0)\omega^* \geq 0, \\ [(I,0,0)\omega^*]^\top [(0,I,0)\omega^*] = 0, \\ (M,-N,-Q)\omega^* - q = 0 \end{cases} \tag{2.1}$$

From the analysis above, we have the following conclusion.

Theorem 2.1 A point $(x^*, y^*) \in R^{2n}$ is a solution of GLCP if and only if there exist $z^* \in R^l$ such that $\omega^* = (x^*, y^*, z^*)$ is a solution of (2.1).

Based on this conclusion, we can convert the GLCP into the following program

$$\begin{array}{ll} \min & \begin{pmatrix} r \\ s \end{pmatrix}^\top \begin{pmatrix} (I,0,0)\omega \\ (0,I,0)\omega \end{pmatrix} \\ s.t. & (I,0,0)\omega \geq 0, (0,I,0)\omega \geq 0, \\ & (M,-N,-Q)\omega - q = 0 \\ & s \geq 0, r \geq 0, r+s = e \end{array} \tag{2.2}$$

and for this problem, we have the following assertion.

Theorem 2.2 A point $(x^*, y^*) \in R^{2n}$ is a solution of GLCP if and only if there exist r^*, s^* such that (ω^*, r^*, s^*) is a solution of (2.2) with the objective varnishing.

Proof. Let $(x^*, y^*) \in R^{2n}$ be a solution of GLCP. Define $s^* \in R^n$ as follows

$$s_i^* = \begin{cases} 1, & i \in I, \\ 0, & j \in J, \end{cases}$$

and r^* be such that $r^* + s^* = e$, where

$$\begin{array}{l} I := \{i \in \{1,2,\cdots,n\} \mid ((I,0,0)\omega^*)_i \leq ((0,I,0)\omega^*)_i\}, \\ J := \{j \in \{1,2,\cdots,n\} \mid ((I,0,0)\omega^*)_j > ((0,I,0)\omega^*)_j\}. \end{array}$$

Combining this with (2.1), we know that (ω^*, r^*, s^*) satisfies the constraints in (2.2) and render zero its objective.

On the other hand, let (ω^*, r^*, s^*) solve (2.2) with the objective varnishing. Then

$${r^*}^\top (I,0,0)\omega^* + {s^*}^\top ((0,I,0)\omega^*) = 0.$$

Since $r^* + s^* = e > 0$, it follow that $((I,0,0)\omega^*)^\top ((0,I,0)\omega^*) = 0$, and ω^* solves the GLCP. □

In the following, we prove that the existence of a vertex solution for (2.2), which will be used to design our algorithm for solving (1.1).

Theorem 2.3 If (2.2) is feasible, then it has a vertex solution.

Proof. Since the quadratic objective function is bounded below by zero on the polyhedral feasible region, it must have a solution, we denote it by $(\bar{\omega}, \bar{r}, \bar{s})$ ([8]). Then problem

$$\begin{array}{ll} \min & \begin{pmatrix} \bar{r} \\ \bar{s} \end{pmatrix}^\top \begin{pmatrix} (I,0,0)\omega \\ (0,I,0)\omega \end{pmatrix} \\ s.t. & (I,0,0)\omega \geq 0, \quad (0,I,0)\omega \geq 0, \\ & (M,-N,-Q)\omega - q = 0 \end{array}$$

has $\hat{\omega}$ such that a vertex $((I,0,0)\hat{\omega}, (0,I,0)\hat{\omega})$ of its feasible region as solution ([9]), and

$$\begin{pmatrix} \bar{r} \\ \bar{s} \end{pmatrix}^\top \begin{pmatrix} (I,0,0)\bar{\omega} \\ (0,I,0)\bar{\omega} \end{pmatrix} = \begin{pmatrix} \bar{r} \\ \bar{s} \end{pmatrix}^\top \begin{pmatrix} (I,0,0)\hat{\omega} \\ (0,I,0)\hat{\omega} \end{pmatrix}$$

Similarly, the linear program

$$\begin{array}{ll} \min & \begin{pmatrix} r \\ s \end{pmatrix}^\top \begin{pmatrix} (I,0,0)\hat{\omega} \\ (0,I,0)\hat{\omega} \end{pmatrix} \\ s.t. & s \geq 0, r \geq 0, r+s=e \end{array}$$

has a vertex $(\hat{r}, \hat{s})$ of its feasible region as solution and

$$\begin{pmatrix} \hat{r} \\ \hat{s} \end{pmatrix}^\top \begin{pmatrix} (I,0,0)\hat{\omega} \\ (0,I,0)\hat{\omega} \end{pmatrix} = \begin{pmatrix} \bar{r} \\ \bar{s} \end{pmatrix}^\top \begin{pmatrix} (I,0,0)\hat{\omega} \\ (0,I,0)\hat{\omega} \end{pmatrix} = \begin{pmatrix} \bar{r} \\ \bar{s} \end{pmatrix}^\top \begin{pmatrix} (I,0,0)\bar{\omega} \\ (0,I,0)\bar{\omega} \end{pmatrix}.$$

Hence $(\hat{\omega}, \hat{r}, \hat{s})$ is a vertex of feasible region of (2.2) and a vertex solution of (2.2). □

Now, we can give the following equivalent reformulation of (2.2).

$$\begin{array}{ll} \min & [(e-s)^\top (I,0,0) + s^\top (0,I,0)]\omega \\ s.t. & (I,0,0)\omega \geq 0, (0,I,0)\omega \geq 0, \\ & (M,-N,-Q)\omega - q = 0 \end{array} \tag{2.3}$$

where

$$s_i = \begin{cases} 1, & i \in I, \\ 0, & j \in J. \end{cases} \quad r+s=e.$$

$$\begin{array}{l} I := \{i \in \{1,2,\cdots,n\} \mid ((I,0,0)\omega)_i \leq ((0,I,0)\omega)_i\}, \\ J := \{j \in \{1,2,\cdots,n\} \mid ((I,0,0)\omega)_j > ((0,I,0)\omega)_j\}, \end{array}$$

3 Algorithm and Convergence

Now, we formally describe our method for solving the GLCP.

Algorithm 3.1

Initial step: Choose with any feasible $\omega^0 \in R^{n+n+l}$ for (2.3), set $k := 0$.

Iterative step: Compute ω^{k+1} by solving the following linear program

$$\begin{aligned} \min \quad & [(e - s(\omega^k))^\top (I,0,0) + s(\omega^k)^\top (0,I,0)]\omega \\ s.t. \quad & (I,0,0)\omega \geq 0, (0,I,0)\omega \geq 0, \\ & (M,-N,-Q)\omega - q = 0 \end{aligned} \tag{3.1}$$

where $s(\omega^k)$ is defined in (2.3) and such that

$$(((e - s(\omega^k))^\top (I,0,0) + s(\omega^k)^\top ((0,I,0)))(\omega^{k+1} - \omega^k) < 0.$$

Stop when impossible.

In the following, we are in the position to show the global convergence of the Algorithm 3.1.

Theorem 3.1 The sequence $\{\omega^k\}$ generated by Algorithm 3.1 terminates in a finite number of steps at a solution of the GLCP or a point $\bar{\omega}$ such that $(\bar{\omega}, r(\bar{\omega}), s(\bar{\omega}))$ is a stationary point of (2.2), where $s(\bar{\omega})$ be define in (2.3).

Proof. The result of theorem is a direct consequence of [7], Theorem 2.1, where the minimization over (r,s) in (2.3) has been carried out resulting in the objective of (2.3) that does not depend on (r,s). □

Theorem 3.2 A point $\bar{\omega}$ satisfies the following necessary optimality conditions of (2.2) for some $\bar{y}_1, \bar{y}_2 \in R^n_+, \bar{y}_3 \in R^m$ such that

$$\begin{cases} (I,0,0)\bar{\omega} \geq 0, \\ (0,I,0)\bar{\omega} \geq 0, \\ (M,-N,-Q)\bar{\omega} - q = 0 \\ (I,0,0)^\top \bar{y}_1 + (0,I,0)^\top \bar{y}_2 + (M,-N,-Q)^\top \bar{y}_3 \\ \qquad = [(e - s(\bar{\omega}))^\top (I,0,0) + s(\bar{\omega})^\top ((0,I,0)]^\top, \\ \bar{y}_1{}^\top ((I,0,0)\bar{\omega}) + \bar{y}_2{}^\top (0,I,0)\bar{\omega} = 0. \end{cases} \tag{3.2}$$

The stationary point $\bar{\omega}$ solves (2.2) if and only if there exist $\bar{y}_1 \in R^n_+, \bar{y}_2 \in R^n_+, \bar{y}_3 \in R^m$ such that

$$\begin{aligned} &(\bar{\omega}, \bar{y}_1, \bar{y}_2) = (\bar{\omega}, s(\bar{\omega}), e - s(\bar{\omega})), \\ &(M,-N,-Q)^\top \bar{y}_3 = 0 \end{aligned}$$

satisfies (3.2).

Proof. Condition (3.2) can be satisfied by

$$(\bar{\omega}, \bar{y_1}, \bar{y_2}) = (\bar{\omega}, s(\bar{\omega}), e - s(\bar{\omega})),$$
$$(M, -N, -Q)^\top \bar{y_3} = 0$$

when $\bar{\omega}$ is a solution of the GLCP.

Conversely, if

$$(\bar{\omega}, \bar{y_1}, \bar{y_2}) = (\bar{\omega}, s(\bar{\omega}), e - s(\bar{\omega})),$$
$$(M, -N, -Q)^\top \bar{y_3} = 0$$

satisfy conditions (3.2), it follows that $\bar{\omega}$ is feasible for the GLCP and that

$$(e - s(\bar{\omega}))^\top (I, 0, 0)\bar{\omega} + s(\omega^k)^\top ((0, I, 0)\bar{\omega}) = 0.$$

Since $(e - s(\bar{\omega})) + s(\bar{\omega}) = e$, it follows that $[(I, 0, 0)\bar{\omega}]^\top [(0, I, 0)\bar{\omega}] = 0$, and $\bar{\omega}$ solves the GLCP.

□

4 Computational Experiments

In the following, we will implement Algorithm 3.1 in Matlab and run it on a Pentium IV computer. Throughout our computation, **Iter** denotes the average number of iterations, **CT** denotes the computing time, and $\mathbf{f}^*$ represents the number of evaluations for the function f, $f^* = (x^*)^\top (y^*)$ is the final value of f when the algorithm terminates.

Example 4.1 This problem is a linear complementarity problem (LCP) used by Harker and Pang ([10]), in which $G(x) = Nx + q$, where

$$N = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 1 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}.$$

For this problem, Harker and Pang ([10]) used the damped-Newton method (DNA), and Wang ([11]) used the Newton-type method (NTA). The results for the above two methods and several values of the dimensions n are summarized in Table 1. In Table 2, we take initial point $x^0 = (1, 1, \cdots, 1)^\top$, and summarize the results of our algorithm for several values of dimensions n. From Table 1 and Table 2, we can conclude that our algorithm excels the other two methods listed above.

To illustrate the stability of our algorithm, under the initial point x^0 is produced randomly in (0,1), and the dimensions $n = 64$, we use it to solve example 4.1, and the results are listed in Table 3. Table 2 and Table 3 indicate that our algorithm is not sensitive to the change of initial point, thus, we can see Algorithm 3.1 performs well for this problem.

Table 1. Numerical Results by DNA, NTA for Example 4.1

Dimension	8	16	32	64	128
DAN iter	9	20	72	208	> 300
NAT iter	13	12	18	99	99

Table 2. Numerical Results of Our Algorithm for Example 4.1

Dimension	8	16	32	64	128
Iter	1	1	1	1	1
$\mathbf{f}^*$	2.6346e-13	9.5940e-12	1.8891e-10	2.1932e-14	3.6643e-13
CT	0.1710	0.1720	0.2030	0.2810	0.6870

Table 3. Numerical Results of Our Algorithm by Random Initial Point for Example 4.1

Trial	1	2	3	4	5
Iter	1	1	1	1	1
$\mathbf{f}^*(1e-14)$	2.1932	2.1932	2.1932	2.1932	2.1932
CT	0.2810	0.2810	0.2810	0.2970	0.2970
Trial	6	7	8	9	10
Iter	1	1	1	2	1
$\mathbf{f}^*(1e-14)$	2.1932	2.1932	2.1932	2.1932	2.1932
CT	0.2970	0.2970	0.2970	0.3750	0.2970

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