International Mathematical Forum, 3, 2008, no. 13, 639 - 644

Completeness of Lorentz Metric on 3-Dimensional Heisenberg Group

Essin Turhan

Fırat Universty, Department of Mathematics
23119, Elazığ, Turkey
eturhan@firat.edu.tr

Abstract

We consider three-dimensional Heisenberg group which is given with possible three situations for Lorentz metrics and obtain completeness of Lorentz metrics on 3-dimensional Heisenberg group.

Mathematics Subject Classifications: 53C50, 22E99

Keywords: Heisenberg group, Lorentz metric, completeness of Lorentz metric

Introduction:

It is shown that we can define three metrics on three dimensional Heisenberg group in [7] . Then in [6] , Rahmani showed that one of the three metrics is flat and he obtain some geodesics using the Euler-Lagrange equations. We know from [5] that a Lie group with left invariant metricis flat if and only if the associated Lie algebra splits as an orthogonal direct *sum* $\boldsymbol{b} \oplus \boldsymbol{u}$ where $\boldsymbol{b}$ is a commutative subalgebra, $\boldsymbol{u}$ is a commutative ideal.

It is known that every left invariant pseudo-Riemannian metric on 2-step Lie group is complete. And then in [2] it is studied that completeness for the left invariant Lorentz metrics on some Lie groups which are a special class of solvable Lie groups. Guediri showed that for some groups every left invariant Lorentz metrics are non-complete.

In this paper, we study geodesicaly completeness of Lorentz metric on 3-dimensional Heisenberg group.

Preliminaries:

Definition: A curve $\gamma : I \longrightarrow H_3$ is inextendible if there exist no curve $\gamma' : I' \longrightarrow H_3$ such that $I' \subset I$ and $\gamma(p) = \gamma'(p)$ for all $p \in I$.

Definition: The curve $\gamma : I \longrightarrow H_3$ is an incomplete geodesic if γ is an inextendible geodesic and $I \neq \mathbb{R}$. We say that 3-dimensional Heisenberg group is geodesically incomplete if it contains an incomplete geodesic. And H_3 is geodes,cly complete if it isn't geodesically incomplete.

Lemma: Let H_3 be a 3-dimensional Heisenberg group with a left invariant Lorentz metric induced by a nondegenerate inner product $\langle , \rangle$ on the $\boldsymbol{h}_3$. For any C^1-curve in H_3, $\gamma : \begin{array}{ccc} I & \longrightarrow & H_3 \\ t & \longrightarrow & \gamma(t) \end{array}$ associate the curve $L^{-1}_{\gamma(t)^*}\gamma'(t)$ in h_3.
Then the curve of $\boldsymbol{h}_3$ associated to geodesics of H_3 are sollutions of the equation

$$X' = ad^*_X X. \tag{1}$$

The Heisenberg group H_{2k+1} admits a flat left invariant Lorentz metric if and only if $k = 1$ It is proved that

$$g_3 = dx^2 + (xdy + dz)^2 - ((1 - x)\,dy - dz)^2$$

is flat by N. Rahmani.

Theorem: The Heisenberg group H_3 with left invariant metric is flat if and only if the associated Lie algebra $\boldsymbol{h}_3$ splits an orthogonal direct *sum* $\boldsymbol{b} \oplus \boldsymbol{u}$ where $\boldsymbol{b}$ is a commutative subalgebra, $\boldsymbol{u}$ is a commutative ideal and where the linear transformation $ad\,(b)$ is skew-adjoint for every $b \in \boldsymbol{b}$.

Let g be a Lorentz metric one of the left invariant Lorentz metrics g_1, g_2, g_3. If $g\,(,)$ is nondegenerate, $e_0 \notin \boldsymbol{u}$ such that

$$g\,(e_0, \boldsymbol{u}) = 0, \quad \boldsymbol{h}_3 = \mathbb{R}e_0 \oplus \boldsymbol{u}.$$

We can obtain

$$ad^*_{e_0} e_0 = 0 \ \text{ and } \ \forall y \in \boldsymbol{u} \ \ ad^*_{e_0} y \in \boldsymbol{u}.$$

So, (1) equation is

$$\begin{cases} \dot{x}_1 = -\dfrac{\langle [e_0, x]\,, x\rangle}{\langle e_0, e_0\rangle} = -\dfrac{\langle Sx, x\rangle}{\langle e_0, e_0\rangle} \\ \dot{x} = x_1 \left(ad^*_{e_0} x\right). \end{cases}$$

Here $L^{-1}_{\gamma(t)^*}\gamma'(t) = x_1 e_0 + x$, $x \in \boldsymbol{u}$ and $S = \frac{1}{2}\,(ad\ e_0 + ad^* e_0)\,.$

It is found out that if $(G, \langle , \rangle)$ is a flat Lorentz 2-step nilpotent Lie group, the restriction of $\langle , \rangle$ to the of $\boldsymbol{b}$ is degenerate by [3].

Completeness of Lorentz Metric:

Theorem: Let (H_3, g_1) is given with the Lorentz metric

$$g_1 = -dx^2 + dy^2 + (xdy + dz)^2 .$$

The left invariant Lorentz metric g_1 is geodesically complete.

Proof: Let our basis be

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x}$$

and we have

$$g_1(e_1, e_1) = g_1(e_2, e_2) = -g_1(e_3, e_3) = 1.$$

We can find easily on the basis $\{e_2, e_3\}$ nontrivial adjoint maps as

$$ad^*_{e_2} e_1 = -e_3, \quad ad^*_{e_3} e_1 = -e_2.$$

The ideal $\boldsymbol{u}$ is genereted by e_1 and is non-null. So we may choose e_2, $e_3 \notin \boldsymbol{u}$ such that

$$g_1(e_2, \boldsymbol{u}) \underset{\boldsymbol{u}}{|} = 0, \quad g_1(e_3, \boldsymbol{u}) \underset{\boldsymbol{u}}{|} = 0 \quad \text{and} \quad \boldsymbol{h}_3 = \mathbb{R}e_0 \oplus \boldsymbol{u}.$$

With this choice we say $\{e_1,\ e_2,\ e_3\}$ is an orhogonal basis of $\boldsymbol{h}_3$ and we obtain

$$[e_2, e_3] = e_1, \quad [e_3, e_1] = 0, \quad [e_2, e_1] = 0.$$

Then, we get easily

$$ad^* e_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

If we take $L^{-1}_{\gamma(t)^*} \gamma'(t) = x_1 e_1 + x_2 e_2 + x_3 e_3$, our (1) equations are

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= -x_1 x_3 \\ \dot{x}_3 &= -x_1 x_2. \end{aligned}$$

So the metric g_1 is geodesically complete.

Theorem: Let (H_3, g_2) is given with the Lorentz metric

$$g_2 = dx^2 + dy^2 - (xdy + dz)^2 .$$

The left invariant Lorentz metric g_2 is geodesically complete.

Proof: Let our basis be

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

and we have

$$g_2 (e_1, e_1) = g_2 (e_2, e_2) = -g_2 (e_3, e_3) = 1.$$

We can find easily on the basis $\{e_1, e_2\}$ nontrivial adjoint maps as

$$ad^*_{e_2} e_3 = -e_1, \quad ad^*_{e_1} e_3 = e_2.$$

The ideal $\boldsymbol{u}$ is genereted by e_3 and is non-null. So we may choose e_1, $e_2 \notin \boldsymbol{u}$ such that

$$g_1 (e_1, \boldsymbol{u}) \underset{\boldsymbol{u}}{|}= 0, \quad g_1 (e_2, \boldsymbol{u}) \underset{\boldsymbol{u}}{|}= 0 \quad \text{and} \quad \boldsymbol{h}_3 = \mathbb{R}e_0 \oplus \boldsymbol{u}.$$

With this choice we say $\{e_1,\ e_2,\ e_3\}$ is an orhogonal basis of $\boldsymbol{h}_3$ and we obtain

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = 0, \quad [e_2, e_3] = 0.$$

Then, we get easily

$$ad^* e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

If we take $L^{-1}_{\gamma(t)^*}\gamma'(t) = x_1e_1 + x_2e_2 + x_3e_3$, our (1) equations are

$$\begin{aligned} \dot{x}_1 &= x_1x_3 \\ \dot{x}_2 &= -x_2x_3 \\ \dot{x}_3 &= 0. \end{aligned}$$

So the metric g_2 is complete.

Theorem: Let (H_3, g_3) is given with the Lorentz metric

$$g_3 = dx^2 + (xdy + dz)^2 - ((1 - x)\,dy - dz)^2.$$

The left invariant Lorentz metric g_3 is geodesically complete.

Proof: It is proved that g_3 is flat by Rahmani. Since g_3 is flat, the ideal $\boldsymbol{u}$ is degenerate. We can choose a basis

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + (1 - x)\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$$

and we have

$$g_3(e_1,e_1) = g_3(e_2,e_2) = -g_3(e_3,e_3) = 1.$$

The ideal $\boldsymbol{u}$ is genereted by $e_2 - e_3$ and is null. So we can choose a new basis $\left\{ x = \frac{1}{\sqrt{2}}(e_2 - e_3),\ y = \frac{1}{\sqrt{2}}(e_3 + e_2),\ z = e_1 \right\}$.With this basis we can obtain

$$[y,z] = 2x = v$$

and

$$g_3(x,y)\underset{\boldsymbol{u}}{|} = 1 \text{ and } g_3(z,z)\underset{\boldsymbol{u}}{|} = 1.$$

We can get

$$ad_v^* y = z, \quad ad_z^* y = -x$$

and

$$ad^* y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

If we take $L^{-1}_{\gamma(t)^*}\gamma'(t) = x_1 v + x_2 y + x_3 z$, our (1) equations are

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= x_1 x_3 \\ \dot{x}_3 &= -x_1 x_2. \end{aligned}$$

So the metric g_3 is complete.

Corollary: Let (H_3, g) be 3-dimensional Heisenberg group, g is left invariant Lorentz metric one of the g_1, g_2, g_3.on H_3. g is geodesically complete on 3-dimensional Heisenberg group.

References

[1] Cordero, A., L., Parker, E., P., "Pseudoriemannian 2-step Nilpotent Lie Groups." 2000.

[2] Guediri, M., "On Completeness of Left Invariant Lorentz Metrics on Solvable Lie Groups", Revista Mathematica, 1996.

[3] Guediri,. M., "Lorentz Geometry of 2-step Nilpotent Lie Groups", Geometrae Dedicata, 2003.

[4] Guediri, M., "On the Nonexistence of Closed Timelike Geodesics in Flat Lorentz 2-step Nilmanifolds", Transations of the American Mathematical Society, 2002.

[5] Milnor, J., Curvatures of Left-Invariant Metrics on Lie Groups, Advances in Mathematics 21, 293-329, 1976.

[6] Rahmani, N., Rahmani, S., Lorentzian Geometry of the Heisenberg Group, Geometriae Dedicata, 133-140, 2006.

[7] Rahmani, S., Metriqus de Lorentz sur les groupes de Lie unimodulaires, de dimension trois, Journal of Geometry and Physics 9, 295-302, 1991.

Received: October 3, 2007