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# On the Frenet Frame and a Characterization of Space-like Involute-Evolute Curve Couple in Minkowski Space-time

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**Abstract**

This paper aims to show that Frenet apparatus of an evolute curve can be formed according to apparatus of involute curve and to present there are no inclined evolutes in Minkowski space-time.

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**Keywords:** Minkowski Space-time, Involute-evolute Curve Couple

## 1 Introduction

It is safe to report that the many important results in the theory of the curves in $E^3$ were initiated by G. Monge; and G. Darboux pionnered the moving frame idea. Thereafter, Frenet defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry. At the begining of the twentieth century, A.Einstein's theory opened a door of use of new geometries. One of them, Minkowski space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold, was introduced and some of classical differential geometry topics have been treated by the researchers.

In [7] authors defined a vector product which is similar to $E^3$ and they gave a method to calculate Frenet apparatus of a space-like and a time-like curve in Minkowski space-time. C. Huygens, who is also known for his works in optics, discovered involutes while trying to build a more accurate clock. And, in particular, involute-evolute curve couple, inclined curves and $W$-curves are well known concepts in the classical differential geometry (see [6] and [5]). Nevertheless relations Frenet apparatus of involute-evolute curve couple in the space $E^3$ are given in [3].

In the present study, space-like involute-evolute curve couple in Minkowski space-time is defined, and Frenet apparatus of this curve couple are described. Thus, this classical topic is extended to the Minkowski space-time. We hope these results will be helpful to mathematicians who are specialized on mathematical modelling.

## 2 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space $E_1^4$ are briefly presented (A more complete elementary treatment can be found in [1]).

Minkowski space-time $E_1^4$ is an Euclidean space $E^4$ provided with the standart flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2, \tag{1}$$

where $(x_1, x_2, x_3, x_4)$ is a rectangular coordinate system in $E_1^4$.

Since $g$ is an indefinite metric, recall that a vector $v \in E_1^4$ can have one of the three causal characters; it can be space-like if $g(v, v) > 0$ or $v = 0$, time-like if $g(v, v) < 0$ and null (light-like) if $g(v, v)$=0 and $v \neq 0$. Similary, an arbitrary curve $\alpha = \alpha(s)$ in $E_1^4$ can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector $v$ is given by $\|v\| = \sqrt{|g(v, v)|}$. Therefore, $v$ is a unit vector if $g(v, v) = \pm 1$. Next, vectors $v, w$ in $E_1^4$ are said to be orthogonal if $g(v, w) = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$. Let $\vartheta = \vartheta(s)$ be a curve in $E_1^4$. If tangent vector field of this curve is forming a constant angle with a constant vector field $U$, then this curve is called an inclined curve. Recall that $W$-curve is a regular curve which has constant curvatures. Now let us define involute-evolute curve couple. Let $\varphi$ and $\delta$ be space-like curves in $E_1^4$. $\varphi$ is an involute of $\delta$ if $\varphi$ lies on the tangent line to $\delta$ at $\delta(s_0)$ and the tangents to $\delta$ and $\varphi$ at $\delta(s_0)$ and $\varphi$ are perpendicular for each $s_0$. $\varphi$ is an evolute of $\delta$ if $\delta$ is an involute of $\varphi$. And this curve couple defined by $\varphi = \delta + \lambda T$.

Denote by $\{T(s), N(s), B(s), E(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space $E_1^4$. Then $T, N, B, E$ are, respectively, the tangent, the principal normal, the binormal and the trinormal vector fields. Space-like or time-like curve $\alpha(s)$ is said to be parametrized by arclength function $s$, if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Let $\alpha(s)$ be a curve in the space-time $E_1^4$, parametrized by arclength function $s$. Then for the curve $\alpha$ the following Frenet equations are given in [4] as follows:

Let $\alpha$ be a space-like curve. Then $T$ is space-like vector, so depending on the causal character of the principal normal vector $N$, subcases are written as follows:

Case 1. $N$ is space-like: Then we will distinguish subcases according to causal character of the binormal $B$.

Case 1.1. $B$ is space-like.In this case the Frenet formulae is read

$$\begin{bmatrix} T' \\ N' \\ B' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & \sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ E \end{bmatrix} \tag{2}$$

where $T, N, B$ and $E$ are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(N,N) = g(B,B) = 1,\ g(E,E) = -1.$$

Case 1.2. $B$ is time-like.The Frenet formulae has the form

$$\begin{bmatrix} T' \\ N' \\ B' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & \tau & 0 & \sigma \\ 0 & 0 & \sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ E \end{bmatrix} \tag{3}$$

where $T, N, B$ and $E$ are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(N,N) = g(E,E) = 1,\ g(B,B) = -1.$$

Case 2. $N$ is time-like. In this case the Frenet equations can be written

$$\begin{bmatrix} T' \\ N' \\ B' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & \tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ E \end{bmatrix} \tag{4}$$

where $T, N, B$ and $E$ are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(B,B) = g(E,E) = 1,\ g(N,N) = -1.$$

Here $\kappa, \tau$ and $\sigma$ are, respectively, first, second and third curvature of curve $\alpha$. In the same space, in [7] authors defined a vector product and gave a method to establish the Frenet frame for an arbitrary curve by following definition and theorem:

Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors in $E_1^4$. The vector product in Minkowski space-time $E_1^4$ is defined by the determinant

$$a \wedge b \wedge c = -\begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \tag{5}$$

where $e_1, e_2, e_3$ and $e_4$ are mutually orthogonal vectors (coordinate direction vectors) satisfying equations

$$e_1 \wedge e_2 \wedge e_3 = e_4\ ,\ e_2 \wedge e_3 \wedge e_4 = e_1\ ,\ e_3 \wedge e_4 \wedge e_1 = e_2\ ,\ e_4 \wedge e_1 \wedge e_2 = -e_3.$$

**Theorem 2.1** *Let $\alpha = \alpha(t)$ be an arbitrary space-like curve in Minkowski space-time $E_1^4$ with space-like tangent, principal normal and binormal and time-like trinormal vector fields. The Frenet apparatus of $\alpha$ can be written as follows;*

$$T = \frac{\alpha'}{\|\alpha'\|}, \tag{6}$$

$$N = \frac{\|\alpha'\|^2 .\alpha'' - g(\alpha', \alpha'').\alpha'}{\left\| \|\alpha'\|^2 .\alpha'' - g(\alpha', \alpha'').\alpha' \right\|}, \tag{7}$$

$$B = \mu N \wedge T \wedge E, \tag{8}$$

$$E = \mu \frac{T \wedge N \wedge \alpha'''}{\|T \wedge N \wedge \alpha'''\|}, \tag{9}$$

$$\kappa = \frac{\left\| \|\alpha'\|^2 .\alpha'' - g(\alpha', \alpha'').\alpha' \right\|}{\|\alpha'\|^4} \tag{10}$$

$$\tau = \frac{\|T \wedge N \wedge \alpha'''\| . \|\alpha'\|}{\left\| \|\alpha'\|^2 .\alpha'' - g(\alpha', \alpha'').\alpha' \right\|} \tag{11}$$

*and*

$$\sigma = \frac{g(\alpha^{(IV)}, E)}{\|T \wedge N \wedge \alpha'''\| . \|\alpha'\|}. \tag{12}$$

In this work, we shall study on space-like curves with equations (2).

# 3 The Frenet apparatus of an involute-evolute curve couple in $\mathbf{E}_1^4$

**Theorem 3.1** *Let $\varphi$ and $\delta$ be unit speed space-like curves and $\varphi$ be an evolute of $\delta$. The Frenet apparatus of $\varphi$ $(\{T_\varphi, N_\varphi, B_\varphi, E_\varphi, \kappa_\varphi, \tau_\varphi, \sigma_\varphi\})$ can be formed according to Frenet apparatus of $\delta$ $(\{T, N, B, E, \kappa, \tau, \sigma\})$.*

**Proof.** From definition of involute-evolute curve couple we write

$$\varphi = \delta + \lambda T. \tag{13}$$

Differentiating both sides of (13) respect to $s$ we obtain

$$\frac{d\varphi}{ds_\varphi}\frac{ds_\varphi}{ds} = T + \frac{d\lambda}{ds}T + \lambda T'. \tag{14}$$

Recalling definition of involute and evolute we can say $T_\beta \perp T$ . Hence we have

$$1 + \frac{d\lambda}{ds} = 0. \tag{15}$$

And we easily find $\lambda = c - s$ where $c$ is constant. Rewriting (13) we have

$$\varphi = \delta + (c - s)T. \tag{16}$$

Differentiating both sides of this expression respect to $s$ we have following equation

$$T_\varphi \frac{ds_\varphi}{ds} = (c - s)\kappa N. \tag{17}$$

Taking the norm of both sides we get

$$T_\varphi = N \tag{18}$$

and

$$\frac{ds_\varphi}{ds} = (c - s)\kappa. \tag{19}$$

Now we will differentiate of (16) four times respect to $s$. From (17), we know $\varphi' = (c - s)\kappa N$. Then the others have the form

$$\varphi'' = -(c - s)\kappa^2 T + [-\kappa + (c - s)\kappa'] N + (c - s)\kappa\tau B. \tag{20}$$

Let us denote $\varphi'''$ and $\varphi^{(IV)}$ with following equations.

$$\varphi''' = u_1 T + u_2 N + u_3 B + u_4 E. \tag{21}$$

$$\varphi^{(IV)} = v_1 T + v_2 N + v_3 B + v_4 E. \tag{22}$$

Considering (7) we calculate and denote

$$\|\varphi'\|^2 .\varphi'' - g(\varphi', \varphi'').\varphi' = l_1 T + l_2 N + l_3 B + l_4 E. \tag{23}$$

Thus, we have principal normal and first curvature of $\varphi$ with the equations

$$N_\varphi = \frac{l_1 T + l_2 N + l_3 B + l_4 E}{\sqrt{-l_1^2 + l_2^2 + l_3^2 + l_4^2}} \tag{24}$$

and

$$\kappa_\varphi = \frac{\sqrt{-l_1^2 + l_2^2 + l_3^2 + l_4^2}}{[(c-s)\kappa]^4}. \tag{25}$$

Using exterior product, we have $T_\varphi \wedge N_\varphi \wedge \varphi'''$ as

$$= \frac{1}{A}\left[(l_3 u_4 - l_4 u_3)T + (l_1 u_4 - l_4 u_1)B + (l_3 u_1 - l_1 u_3)E\right], \tag{26}$$

where $A = \sqrt{-l_1^2 + l_2^2 + l_3^2 + l_4^2}$. Thus, we have trinormal of $\varphi$

$$E_\varphi = \mu \frac{[(l_3 u_4 - l_4 u_3)T + (l_1 u_4 - l_4 u_1)B + (l_3 u_1 - l_1 u_3)E]}{\sqrt{-(l_3 u_4 - l_4 u_3)^2 + (l_1 u_4 - l_4 u_1)^2 + (l_3 u_1 - l_1 u_3)^2}}. \tag{27}$$

And easily we have second and third curvatures as follows:

$$\tau_\varphi = (c-s)\kappa.\sqrt{\frac{-(l_3 u_4 - l_4 u_3)^2 + (l_1 u_4 - l_4 u_1)^2 + (l_3 u_1 - l_1 u_3)^2}{-l_1^2 + l_2^2 + l_3^2 + l_4^2}}, \tag{28}$$

$$\sigma_\varphi = \frac{v_4}{(c-s)\kappa.\sqrt{-(l_3 u_4 - l_4 u_3)^2 + (l_1 u_4 - l_4 u_1)^2 + (l_3 u_1 - l_1 u_3)^2}}. \tag{29}$$

Finally, the exterior product $N_\varphi \wedge T_\varphi \wedge E_\varphi$ gives us binormal vector as

$$B_\varphi = -\frac{\mu}{F} \frac{\left\{\begin{array}{c} [l_3(l_3 u_1 - l_1 u_3) - l_4(l_1 u_4 - l_4 u_1)]\,T \\ + [l_1(l_3 u_1 - l_1 u_3) - l_4(l_3 u_4 - l_4 u_3)]\,B \\ + l_3(l_3 u_4 - l_4 u_3) - l_1(l_1 u_4 - l_4 u_1)E \end{array}\right\}}{\sqrt{-(l_3 u_4 - l_4 u_3)^2 + (l_1 u_4 - l_4 u_1)^2 + (l_3 u_1 - l_1 u_3)^2}}, \tag{30}$$

where $F = \sqrt{-l_1^2 + l_2^2 + l_3^2 + l_4^2}$.

# 4 A special characterization of an involute-evolute curve couple in $\mathbf{E}_1^4$

**Theorem 4.1** *Let $\varphi$ and $\delta$ be unit speed space-like curves and $\varphi$ be an evolute of $\delta$ in Minkowski space-time. The evolute $\varphi$ is never an inclined curve in $E_1^4$.*

**Proof.** Considering definition of inclined curves, we write

$$T_\varphi.U = \cos\theta, \tag{31}$$

where $U$ is a constant space-like vector and $\theta$ is constant angle. From (18) we easily have

$$N.U = \cos\theta. \tag{32}$$

Differentiating both sides of (32) we obtain

$$(-\kappa T + \tau B).U = 0. \tag{33}$$

Therefore, $U \perp T$ and $U \perp B$. Let us decompose $U$ as

$$U = t_1 N + t_2 E. \tag{34}$$

One more differentiating of (34) and using Frenet equations we have $t_1 = 0$ and $t_2 = 0$. By this result, we write

$$U = 0. \tag{35}$$

By (31), (35) yields a contradiction. Therefore, evolute $\varphi$ can not be an inclined curve in the space $E_1^4$.

**Remark 4.1** In the case when $\varphi$ is space-like curve within other cases in the space $E_1^4$, there holds theorems which are analogous theorem (3.1) and (4.1).

**Example 4.1** Now, we present a special application of theorem (3.1). Let $\varphi$ and $\delta$ be unit speed space-like curves and $\varphi$ be an evolute of $\delta$. Moreover, let us suppose $\delta$ is a $W-$curve. To calculate Frenet apparatus, first we calculate following derivatives:

$$\varphi' = (c-s)\kappa N. \tag{36}$$

$$\varphi'' = (s-c)\kappa^2 T - \kappa N + (c-s)\kappa\tau B. \tag{37}$$

$$\varphi''' = 2\kappa^2 T + (s-c)\kappa N - (c-s)\kappa\tau^2 N - 2\kappa\tau B + (c-s)\kappa\tau\sigma E. \quad (38)$$

$$\left\{ \begin{array}{c} \varphi^{(IV)} = (c-s)\left[\kappa^2(1+\tau^2)\right]T + \left[\kappa(2\kappa^2+3\tau^2+1)\right]N \\ +(s-c)\left[\kappa\tau(\tau^2-\sigma^2+1)\right]B - 3\kappa\tau\sigma E \end{array} \right\}. \quad (39)$$

And following prosedure in theorem (3.1), we first form

$$\left\|\varphi'\right\|^2 .\varphi'' - g(\varphi',\varphi'').\varphi' = (c-s)^3\kappa^3\left[\kappa T + \tau B\right]. \quad (40)$$

Using (24), we have principal normal of $\varphi$ as

$$N_\varphi = \frac{\kappa T + \tau B}{\sqrt{\tau^2 - \kappa^2}}. \quad (41)$$

Using exterior product we get trinormal of $\varphi$

$$E_\varphi = \mu\left[\frac{(c-s)\tau\sigma T + (c-s)\kappa\sigma B + 4\kappa E}{\sqrt{(c-s)^2(\kappa\sigma)^2(1-\tau^2)+16\kappa^2}}\right]. \quad (42)$$

Considering obtained equations, we easily have the curvatures of $\varphi$ as follows:

$$\kappa_\varphi = \frac{\sqrt{\tau^2-\kappa^2}}{(c-s)\kappa}, \quad (43)$$

$$\tau_\varphi = \kappa\tau\sqrt{\frac{(c-s)^2(\kappa\sigma)^2(1-\tau^2)+16\kappa^2}{\tau^2-\kappa^2}} \quad (44)$$

and

$$\sigma_\varphi = \frac{-3\sigma}{\left[(c-s)\kappa\right]^4\sqrt{(c-s)^2(\kappa\sigma)^2(1-\tau^2)+16\kappa^2}}. \quad (45)$$

And finally exterior product $N_\varphi \wedge T_\varphi \wedge E_\varphi$ gives us binormal of $\varphi$

$$B_\varphi = \mu(c-s)^6\kappa^7\tau\left[4\kappa\tau T + \kappa^2 B + (c-s)\sigma(\tau^2-\sigma^2)E\right]. \quad (46)$$

**Corollary 4.2** *In the case of $\delta$ is a $W-$curve, suffice it to say that evolute $\varphi$ may not be a $W$-curve.*

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