International Mathematical Forum, 3, 2008, no. 16, 783 - 792

On the Characterizations of Inclined Curves in Minkowski Space-Time $\mathbf{E}_1^4$

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Abstract

In this work, necessary and sufficient conditions to be inclined for a space-like and a time-like curve in Minkowski space-time are presented.

Mathematics Subject Classification: 53C50

Keywords: Minkowski Space-time, Inclined Curves

1 Introduction

In the classical differential geometry of curves in E^3, it is well known concept to be inclined if the curve in the case of differentiable (see,[6]). Magden [1] and Hacisalihoglu [3] extended this concept to the space E^4 and E^n and wrote necessary and sufficient conditions, respectively.In another work Ekmekci [5] characterized inclined curves in L^n Lorentzian space by means of harmonic curvatures. Using this formulae Yilmaz [7] gave a theorem which characterizes inclined space-like curves in the space E_1^4.

The aim of this paper to present necessary and sufficient conditions to be inclined for space-like and time-like curves in terms of Frenet equations.

2 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^4 are briefly presented.(A more complete elementary treatment can be found in [2]).

Minkowski space-time E_1^4 is an Euclidean space E^4 provided with the standart flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2, \tag{1}$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4.

Since g is an indefinite metric, recall that a vector $v \in E_1^4$ can have one of the three causal characters; it can be space-like if $g(v, v) > 0$ or $v = 0$, time-like if $g(v, v) < 0$ and null (light-like) if $g(v, v)$=0 and $v \neq 0$. Similary, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector v is given by $\|v\| = \sqrt{|g(v, v)|}$. Therefore, v is a unit vector if $g(v, v) = \pm 1$. Next, vectors v, w in E_1^4 are said to be orthogonal if $g(v, w) = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$.

Let a be a time-like vector in E_1^4. If $g(a, e) > 0$, then a is past-pointing time-like vector; if $g(a, e) < 0$, then a future-pointing time like vector, where $e = (0, 0, 0, 1)$. Let a and b be two time-like vector in E_1^4.If a and b are in the same time cone of E_1^4,there is unique real number $\delta \geq 0$ called the hyperbolic angle between a and b, such that $g(a, b) = -\|a\| \|b\| \cosh \delta$.And if a and b aren't in the same time cone then there is unique real number $\delta \geq 0$ called the hyperbolic angle between a and b, such that $g(a, b) = \|a\| \|b\| \cosh \delta$.

Let $\vartheta = \vartheta(s)$ be a curve in E_1^4. If tangent vector field of this curve is forming a constant angle with a constant vector field U, then this curve is called an inclined curve.

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^4. Then T, N, B_1, B_2 are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Space-like or time-like curve $\alpha(s)$ is said to be parametrized by arclength function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Let $\alpha(s)$ be a curve in the space-time E_1^4, parametrized by arclength function s. Then for the curve α the following Frenet equations are given in [4] :

Case 1 : α is a space-like curve. Then T is space-like vector, so depending on the causal character of the principal normal vector N, subcases are written as follows:

Case 1.1. N is space-like: Then we will distinguish subcases according to causal character of the first binormal B_1.

Case 1.1.1. B_1 is space-like.In this case the Frenet formulae is read

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & \sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{2}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(N,N) = g(B_1,B_1) = 1,\ g(B_2,B_2) = -1.$$

Case 1.1.2. B_1 is time-like.The Frenet formulae has the form

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & \tau & 0 & \sigma \\ 0 & 0 & \sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{3}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(N,N) = g(B_2,B_2) = 1,\ g(B_1,B_1) = -1.$$

Case 1.2.N is time-like. In this case the Frenet equations can be written

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & \tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{4}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(B_1,B_1) = g(B_2,B_2) = 1,\ g(N,N) = -1.$$

Case 2 : α is a time-like curve. Then T is time-like vector. And the Frenet equations have the form

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{5}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$g(N,N) = g(B_1,B_1) = g(B_2,B_2) = 1,\ g(T,T) = -1.$$

Here κ, τ and σ are, respectively, first, second and third curvature of the curve α.

In the same space, Yilmaz (see [7]) gave a formulation about inclined curves with the following theorem.

Theorem 2.1 *Let $\alpha = \alpha(s)$ be a space-like curve in E_1^4 parametrized by arclength. The curve α is an inclined curve if and only if*

$$\frac{\kappa}{\tau} = A\cosh\left(\int_0^s \sigma ds\right) + B\sinh\left(\int_0^s \sigma ds\right), \tag{6}$$

where $\tau \neq 0$ and $\sigma \neq 0$, $A, B \in R$.

3 The Inclined Space-like Curves in $\mathbf{E}_1^4$

Theorem 3.1 *Let $\psi = \psi(s)$ be a space-like curve in E_1^4. ψ is an inclined curve if and only if*

$$\left(\frac{\kappa}{\tau}\right)^2 - \frac{1}{\sigma^2}\left[\frac{d}{ds}\left(\frac{\kappa}{\tau}\right)\right]^2 = 0. \tag{7}$$

Proof. Let $\psi = \psi(s)$ be a space-like curve in E_1^4. From definition of inclined curves we can write

$$T.U = \cos\theta, \tag{8}$$

where U is a space-like constant vector. Differentiating both sides of this equation we have

$$\kappa N.U = 0. \tag{9}$$

Thus, we arrive $N \perp U$. Considering this we can compose U as

$$U = u_1 T + u_2 B_1 + u_3 B_2, \tag{10}$$

where $u_i, 1 \leq i \leq 3$ are arbitrary functions. Differentiating (10) and considering Frenet equations, we have

$$u_1 = c_1 = \text{constant and nonzero},$$

$$u_2 = \tfrac{c_1\kappa}{\tau} = -\tfrac{1}{\sigma}\tfrac{du_3}{ds}$$

and

$$\frac{du_2}{ds} = -\sigma u_3. \tag{11}$$

And from last equations we have $u_3 = -\frac{c_1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau})$.Differentiating u_2 we have second ordered linear differential equation as

$$-\frac{d}{ds}\left(\frac{1}{\sigma}\frac{du_3}{ds}\right) + \sigma u_3 = 0. \tag{12}$$

By using exchange variable $t = \int\limits_0^s \sigma ds$ in (12) we have

$$-\frac{d^2u_3}{dt^2} + u_3 = 0. \tag{13}$$

The general solution of (13) is

$$u_3 = k_1 e^t + k_2 e^{-t} \tag{14}$$

or

$$u_3 = k_1(\cosh t + \sinh t) + k_2(\cosh t - \sinh t), \tag{15}$$

where $k_1$and $k_2 \in R$.

By taking $k_1 + k_2 = A_1$ and $k_1 - k_2 = A_2$, shortly, we have

$$u_3 = A_1 \cosh t + A_2 \sinh t. \tag{16}$$

Replacing variable $t = \int\limits_0^s \sigma ds$ in (16) we obtain

$$u_3 = -\frac{c_1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau}) = A_1 \cosh\left(\int\limits_0^s \sigma ds\right) + A_2 \sinh\left(\int\limits_0^s \sigma ds\right). \tag{17}$$

Considering equations (11) we have

$$u_2 = \frac{c_1\kappa}{\tau} = -A_1 \sinh\left(\int\limits_0^s \sigma ds\right) - A_2 \cosh\left(\int\limits_0^s \sigma ds\right). \tag{18}$$

From last two equations we can calculate two real numbers A_1, A_2 with Cramer method as

$$A_1 = A_2 = -\frac{c_1\kappa}{\tau}\cosh\left(\int\limits_0^s \sigma ds\right) + \frac{c_1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau})\sinh\left(\int\limits_0^s \sigma ds\right). \tag{19}$$

If we calculate $A_2^2 - A_1^2$, we have

$$c_1^2(\frac{\kappa}{\tau})^2 - \frac{c_1^2}{\sigma^2}\left[\frac{d}{ds}(\frac{\kappa}{\tau})\right]^2 = 0. \tag{20}$$

or

$$(\frac{\kappa}{\tau})^2 - \frac{1}{\sigma^2}\left[\frac{d}{ds}(\frac{\kappa}{\tau})\right]^2 = 0. \tag{21}$$

Conversely, let us consider vector given by

$$U = \left\{T + \frac{\kappa}{\tau}B_1 - \frac{1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau})B_2\right\}\cos\theta. \tag{22}$$

Differentiating vector U and considering differential of (21), we easily obtain

$$\frac{dU}{ds} = 0. \tag{23}$$

Thus U is a constant vector and so $\psi(s)$ is an inclined curve in E_1^4.

Corollary 3.2 *Let $\psi = \psi(s)$ be a space-like curve in E_1^4. ψ is an inclined curve if and only if*

$$\sigma\frac{\kappa}{\tau} - \frac{d}{ds}\left[\frac{1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau})\right] = 0. \tag{24}$$

Remark 3.1 In the case when ψ is space-like curve with time-like principal normal in the space E_1^4, there holds theorems which are analogous theorem (3.1).

Now let us solve the equation (24) respect to $\frac{\kappa}{\tau}$.If we use exchange variable $t = \int\limits_0^s \sigma ds$ in (24), we have

$$-\frac{d^2}{dt^2}\left(\frac{\kappa}{\tau}\right) + \left(\frac{\kappa}{\tau}\right) = 0. \tag{25}$$

and so we arrive Yilmaz's [7] equation as

$$\frac{\kappa}{\tau} = L_1\cosh\left(\int\limits_0^s \sigma ds\right) + L_2\sinh\left(\int\limits_0^s \sigma ds\right). \tag{26}$$

where L_1 and L_2 are real numbers.

4 The Inclined Time-like Curves in $\mathbf{E}_1^4$

Theorem 4.1 *Let $\xi = \xi(s)$ be a time-like curve in E_1^4. ξ is an inclined curve if and only if*

$$(\frac{\kappa}{\tau})^2 + \frac{1}{\sigma^2}\left[\frac{d}{ds}(\frac{\kappa}{\tau})\right]^2 = 0. \tag{27}$$

Proof. Let $\xi = \xi(s)$ be a time-like curve in E_1^4.From definition of inclined curves we can write

$$T.U = \eta \cosh\theta, \tag{28}$$

where U is a time-like constant vector and η is taken ± 1 according to character of U. Differentiating both sides of this equation we have

$$\kappa N.U = 0. \tag{29}$$

Thus, we arrive, again, similar to a space-like curve, $N \perp U$. Let us compose U as

$$U = v_1 T + v_2 B_1 + v_3 B_2, \tag{30}$$

where $u_i, 1 \leq i \leq 3$ are arbitrary functions.

Differentiating (30) and considering Frenet equations, we have

$$v_1 = c_1 = \text{constant and nonzero,}$$

$$v_2 = \tfrac{c_1\kappa}{\tau} = -\tfrac{1}{\sigma}\tfrac{dv_3}{ds}$$

and

$$\frac{dv_2}{ds} = \sigma v_3. \tag{31}$$

Last equations imply that $v_3 = \frac{c_1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau})$. Using obtained equations, we have

$$\frac{d}{ds}\left(\frac{1}{\sigma}\frac{dv_3}{ds}\right) + \sigma v_3 = 0. \tag{32}$$

By using exchange variable $t = \int\limits_0^s \sigma ds$ in (32), we get second ordered linear homogenus differential equation as

$$\frac{d^2 v_3}{dt^2} + v_3 = 0. \tag{33}$$

The general solution of (33) is

$$v_3 = m_1 \cos t + m_2 \sin t, \tag{34}$$

where $m_1$and $m_2 \in R$. Rewriting $t = \int\limits_0^s \sigma ds$ in (34), we have

$$v_3 = \frac{c_1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau}) = m_1 \cos\left(\int\limits_0^s \sigma ds\right) + m_2 \sin\left(\int\limits_0^s \sigma ds\right) \tag{35}$$

and

$$v_2 = \frac{c_1 \kappa}{\tau} = m_1 \sin\left(\int\limits_0^s \sigma ds\right) - m_2 \cos\left(\int\limits_0^s \sigma ds\right). \tag{36}$$

(35) and (36) imply that

$$m_1 = \frac{c_1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau}) \cos\left(\int\limits_0^s \sigma ds\right) + \frac{c_1 \kappa}{\tau} \sin\left(\int\limits_0^s \sigma ds\right) \tag{37}$$

and

$$m_2 = -\frac{c_1 \kappa}{\tau} \cos\left(\int\limits_0^s \sigma ds\right) + \frac{c_1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau}) \sin\left(\int\limits_0^s \sigma ds\right). \tag{38}$$

If we calculate $m_2^2 + m_1^2$, we have

$$c_1^2(\frac{\kappa}{\tau})^2 + \frac{c_1^2}{\sigma^2}\left[\frac{d}{ds}(\frac{\kappa}{\tau})\right]^2 = 0 \tag{39}$$

or

$$(\frac{\kappa}{\tau})^2 + \frac{1}{\sigma^2}\left[\frac{d}{ds}(\frac{\kappa}{\tau})\right]^2 = 0. \tag{40}$$

Conversely, let us compose vector

$$U = \left\{T + \frac{\kappa}{\tau}B_1 + \frac{1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau})B_2\right\}\eta \cosh\theta. \tag{41}$$

Differentiating vector U and considering differential of (40), we easily obtain

$$\frac{dU}{ds} = 0. \tag{42}$$

Thus U is a constant vector and so $\xi = \xi(s)$ is an inclined curve in E_1^4.

Corollary 4.2 *Let* $\xi = \xi(s)$ *be a time-like curve in* E_1^4*.* ξ *is an inclined curve if and only if*

$$\sigma\frac{\kappa}{\tau} + \frac{d}{ds}\left[\frac{1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau})\right] = 0. \tag{43}$$

Let us solve this equation respect to $\frac{\kappa}{\tau}$.If we use exchange variable $t = \int\limits_0^s \sigma ds$ in (43), we have

$$\frac{d^2}{dt^2}\left(\frac{\kappa}{\tau}\right) + \left(\frac{\kappa}{\tau}\right) = 0, \tag{44}$$

and so

$$\frac{\kappa}{\tau} = W_1 \cos\left(\int\limits_0^s \sigma ds\right) + W_2 \sin\left(\int\limits_0^s \sigma ds\right), \tag{45}$$

where W_1 and W_2 are real numbers.

Corollary 4.3 *Let* $\xi = \xi(s)$ *be a time-like curve in* E_1^4*.* ξ *is an inclined curve if and only if*

$$\frac{\kappa}{\tau} = W_1 \cos\left(\int\limits_0^s \sigma ds\right) + W_2 \sin\left(\int\limits_0^s \sigma ds\right), \tag{46}$$

where W_1 *and* W_2 *are real numbers.*

ACKNOWLEDGEMENTS.

The second author would like to thank TUBITAK-BIDEB for their financial supports during his Master of Science studies.

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Received: October 20, 2007