International Mathematical Forum, 3, 2008, no. 14, 695 - 701

On the Formalization of the Solution of Fredholm Integral Equations with Degenerate Kernel

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Abstract

In this paper we discuss on the solution of Fredholm integral equation of the second kind

$$\phi(x) = f(x) + \lambda \int_a^b k(x,t)\phi(t)dt \tag{1}$$

with the Degenerate kernel

$$k(x,t) = \sum_{k=1}^{n} a_k(x)b_k(t)$$

and show that the solution of (1) is the form

$$\phi(x) = f(x) + \lambda \sum_{k=1}^{n} C_k a_k(x)$$

so that

$$C_k = \frac{1}{\triangle(\lambda)} \begin{vmatrix} 1-\lambda a_{11} & \cdots & -\lambda a_{1(k-1)} & f_1 & -\lambda a_{1(k+1)} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & \cdots & -\lambda a_{2(k-1)} & f_2 & -\lambda a_{2(k+1)} & \cdots & -\lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\lambda a_{n1} & \cdots & -\lambda a_{n(k-1)} & f_n & -\lambda a_{n(k+1)} & \cdots & 1-\lambda a_{nn} \end{vmatrix}.$$

and

$$\triangle(\lambda)=\begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \cdots & -\lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda a_{n1} & -\lambda a_{n2} & \cdots & 1-\lambda a_{nn} \end{vmatrix} \neq 0.$$

$k=1,2,\cdots,n$ and $a_{km}=\int_a^b a_k(t)b_m(t)dt$.
And if $m=k$, we show that

$$\phi(x)=f(x)+\lambda\sum_{k=1}^{n}\frac{f_k}{1-\lambda a_{kk}}a_k(x)$$

so that

$$\begin{cases} f_k=\int_a^b b_k(t)f(t)dt \\ a_{kk}=\int_a^b a_k(t)b_k(t)dt \end{cases}$$

Mathematics Subject Classification: 45B05, 74H10

Keywords: Integral equations, Degenerate kernel, Adomian decomposition method

1 Introduction

The kernel $k(x,t)$ of a Fredholm integral equation of the second kind is called "Degenerate" if it is of the form:

$$k(x,t)=\sum_{k=1}^{n}a_k(x)b_k(t)$$

We shall consider the functions $a_k(x)$ and $b_k(t)$ $(k=1,2,\cdots,n$)which are continuous in the square ; $a\leq x,t\leq b$; and linearly independent.
Therefore we have:

$$\phi(x)=f(x)+\lambda\sum_{k=1}^{n}a_k(x)\int_a^b b_k(t)\phi(t)dt \tag{2}$$

and if we introduce the notation

$$C_k=\int_a^b b_k(t)\phi(t)dt \tag{3}$$

then

$$\phi(x)=f(x)+\lambda\sum_{k=1}^{n}C_k a_k(x) \tag{4}$$

where C_k are unknown constants.

2 Solution in the general case

With putting the expression (4) in to the integral equation (2) , we get

$$\sum_{m=1}^{n}\{C_m - \int_a^b b_m(t)[f(t)+\lambda\sum_{k=1}^{n}C_k a_k(t)]dt\}a_m(x)=0$$

and

$$C_m - \lambda\sum_{k=1}^{n}C_k\int_a^b a_k(t)b_m(t)dt=\int_a^b b_m(t)f(t)dt$$

We introduce the notations

$$a_{km}=\int_a^b a_k(t)b_m(t)dt$$

$$f_m=\int_a^b b_m(t)f(t)dt$$

then we will have

$$C_m - \lambda\sum_{k=1}^{n}a_{km}C_m = f_m$$

$$m=1,2,\cdots,n$$

then

$$\begin{cases}(1-\lambda a_{11})C_1-\lambda a_{12}C_2-\cdots-\lambda a_{1n}C_n=f_1\\ -\lambda a_{21}C_1+(1-\lambda a_{22})C_2-\cdots-\lambda a_{2n}C_n=f_2\\ \cdots\\ -\lambda a_{n1}C_1-\lambda a_{n2}C_2-\cdots+(1-\lambda a_{nn})C_n=f_n\end{cases} \tag{5}$$

To solve the unknowns C_k ; $k=1,2,\cdots,n$; we should have

$$\triangle(\lambda)=\begin{vmatrix}1-\lambda a_{11} & -\lambda a_{12} & \cdots & -\lambda a_{1n}\\ -\lambda a_{21} & 1-\lambda a_{22} & \cdots & -\lambda a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ -\lambda a_{n1} & -\lambda a_{n2} & \cdots & 1-\lambda a_{nn}\end{vmatrix}\neq 0. \tag{6}$$

Then C_k are obtained from the Cramer's formula

$$C_k=\frac{1}{\triangle(\lambda)}\begin{vmatrix}1-\lambda a_{11} & \cdots & -\lambda a_{1(k-1)} & f_1 & -\lambda a_{1(k+1)} & \cdots & -\lambda a_{1n}\\ -\lambda a_{21} & \cdots & -\lambda a_{2(k-1)} & f_2 & -\lambda a_{2(k+1)} & \cdots & -\lambda a_{2n}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ -\lambda a_{n1} & \cdots & -\lambda a_{n(k-1)} & f_n & -\lambda a_{n(k+1)} & \cdots & 1-\lambda a_{nn}\end{vmatrix}. \tag{7}$$

$$k=1,2,\cdots,n$$

Then the solution of the Fredholm integral equation of the second kind (1) is the function $\phi(x)$ defined by the equality

$$\phi(x) = f(x) + \lambda \sum_{k=1}^{n} C_k a_k(x)$$

where the coefficients C_k are determind from formulas (7). Then:

Theorem 1: *The solution of Fredholm integral equation of the second kind*

$$\phi(x) = f(x) + \lambda \int_a^b k(x,t)\phi(t)dt$$

with the Degenerate kernel $k(x,t) = \sum_{k=1}^{n} a_k(x) b_k(t)$*,so that* $a_k(x)$ *and* $b_k(t)$ $(k = 1, 2, \cdots, n$ *)which are continuous in the square ;* $a \leq x, t \leq b$ *; and linearly independent, is the form*

$$\phi(x) = f(x) + \lambda \sum_{k=1}^{n} C_k a_k(x)$$

where the coefficients $C_k, k = 1, 2, \cdots, n$ *are determined from* (7).

Example:

$$\phi(x) = \cos x + 2 \int_0^{2\pi} (\sum_{k=1}^{n} \sin kx \cos kt)\phi(t)dt$$

Solving by formula:

$$\phi(x) = \cos x + 2 \int_0^{2\pi} (\sum_{k=1}^{n} \sin kx \cos kt)\phi(t)dt$$

$$= \cos x + 2 \sum_{k=1}^{n} \sin kx \int_0^{2\pi} \cos kt \phi(t)dt$$

then

$$\phi(x) = \cos x + 2 \sum_{k=1}^{n} C_k \sin kx$$

where

$$C_k = \int_0^{2\pi} \cos kt \phi(t)dt$$

then

$$C_k = \int_0^{2\pi} \cos kt(\cos t + 2 \sum_{k=1}^{n} C_k \sin kt)dt$$

then

$$\begin{cases} C_1 = \pi \\ C_k = 0 \quad k = 2, 3, \cdots, n \end{cases}$$

Then the solution of the Fredholm integral equation is

$$\phi(x) = \cos(x) + 2\pi \sin(x)^1$$

3 Remark

Now, if we consider that

$$\begin{cases} a_{km} = 0 & k \neq m \\ a_{km} \neq 0 & k = m \end{cases} \tag{8}$$

Then

$$\triangle(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & 0 & \cdots & 0 \\ 0 & 1-\lambda a_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1-\lambda a_{nn} \end{vmatrix}. \tag{9}$$

or

$$\triangle(\lambda) = \prod_{k=1}^{n} (1 - \lambda a_{kk})$$

If $\triangle(\lambda) \neq 0$ or $\lambda \neq \frac{1}{a_{kk}}, k = 1, 2, \cdots, n$, then the system (5) has a unique solution

$$\begin{cases} C_k = \frac{f_k}{1-\lambda a_{kk}} \\ k = 1, 2, \cdots, n \end{cases}$$

In this case the solution of the Fredholm integral equation of the second kind is the function ϕ, defined by the equality

$$\phi(x) = f(x) + \lambda \sum_{k=1}^{n} \frac{f_k}{1 - \lambda a_{kk}} a_k(x)$$

therefore we have:

Theorem 2: *The solution of the Fredholm integral equation of the second kind*

$$\phi(x) = f(x) + \lambda \int_a^b k(x,t)\phi(t)dt$$

defined by the equality

$$\phi(x) = f(x) + \lambda \sum_{k=1}^{n} \frac{f_k}{1 - \lambda a_{kk}} a_k(x)$$

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[1] By Adomian decomposition method the solution be $\phi(x) = \cos(x) + 2\pi \sin(x)$

if

$$\begin{cases} k(x,t)=\sum_{k=1}^{n} a_k(x)b_k(t) \\ a_{km}=\int_a^b a_k(t)b_m(t)dt \\ f_m=\int_a^b b_m(t)f(t)dt \end{cases}$$

and if

$$\begin{cases} a_{km}=0 \quad k\neq m \\ a_{km}=0 \quad k=m \end{cases}$$

Example:

$$\phi(x)=\cos 3x+\lambda\int_0^{\pi}\cos(x+t)\phi(t)dt$$

Solving by formula: we have

$$\phi(x)=\cos 3x+\lambda C_1\cos x-\lambda C_2\sin x$$

where

$$C_1=\int_0^{\pi}\cos t\phi(t)dt$$

$$C_2=\int_0^{\pi}\sin t\phi(t)dt$$

then

$$\triangle(\lambda)=\begin{vmatrix} 1-\lambda\frac{\pi}{2} & 0 \\ 0 & 1+\lambda\frac{\pi}{2} \end{vmatrix}$$

If $\lambda\neq 0$, then we have

$$f_1=\int_0^{\pi}\cos t\cos 3tdt=0$$

$$f_2=\int_0^{\pi}\sin t\cos 3tdt=0$$

$$a_{11}=\int_0^{\pi}\cos^2 tdt=\frac{\pi}{2}$$

$$a_{22}=-\int_0^{\pi}\sin^2 tdt=-\frac{\pi}{2}$$

Then the solution of the Fredholm integral equation is

$$(\phi(x)=\cos 3x)^2$$

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[2] By Adomian decomposition method the solution be $\phi(x)=\cos 3x$

References

[1] G. Adomian : ”Nonlinear Stochastic Systems and Application to Physics” , Kluwer Academic Press, 1989.

[2] G. Adomian : ”Solving Frontier Problems of Physics:The Decomposition Method”, Kluwer Academic Priss, 1994.

[3] H. H. Kagiwada, R. Kalaba : ”An initial-value theory for Fredholm integral equations with semidegenerate kernel”, JACM volume 17(p:412-419), 1970.

[4] H. Kaneko, Y. Xu : ”Degenerate kernel method for Hammerstein equations”, Math. comp., 56 (p:141-148), 1991.

[5] H. Kaneko, R. Noren and Y. Xu : ”Regularity of the solution of Hammerstein equations with weakly singular kernels”, Equations Operator Theory, 13(p:660-670), 1990.

Received: October 23, 2007