International Mathematical Forum, 3, 2008, no. 14, 671 - 676

Estimation of Unique Variances Using G-inverse Matrix in Factor Analysis

Seval Süzülmüş

Osmaniye Korkut Ata University
Vocational High School of Osmaniye
Osmaniye, Turkey
suzulmus@cu.edu.tr

Sadullah Sakallıoğlu

Çukurova University
Faculty Science and Letters
Department of Statistics
01330 Adana, Turkey

Abstract

The problem of estimation of parameters in factor analysis is one of the important phase and has attracted several researchers. In all methods, when identifying of parameters, Σ (variance-covariance matrix) is positive definite. In a few studies, when Σ is not positive definite generalized inverse (g-inverse) is used [3,6]. So that, when the population variance-covariance (correlation) matrix is non-negative definite we define the estimators of unique variances in factor analysis.

Mathematics Subject Classification: 15A09, 62H25

Keywords: Factor analysis, unique variance, g-inverse

1 Introduction

The observable random vector $\mathbf{x}$, with p components, has mean μ and covariance matrix $\mathbf{\Sigma}$. Under the factor analysis model $\mathbf{x}$ can be written in the form

$$\mathbf{x} = \mu + \mathbf{\Lambda f} + \mathbf{e}$$

where $\mathbf{\Lambda} = (\lambda_{ij})$ is a pxk matrix of factor loadings; $\mathbf{f} = (f_1, f_2, \ldots, f_k)'$ and $\mathbf{e} = (e_1, e_2, \ldots, e_p)'$ are unobservable random vectors. The elements of $\mathbf{f}$ and $\mathbf{e}$ are called the common factors and the unique factors, respectively. We assume that the means of the elements of $\mathbf{f}$ and $\mathbf{e}$ are zero and that $E(\mathbf{ff}') = I_k$ and $E(\mathbf{ee}') = \Psi$, where I_k is the identity matrix of order k and $\mathbf{\Psi}$ is a diagonal matrix, of which the diagonal elements $\Psi_j\ (> 0)$ are called the unique variances. It will furthermore be assumed that $E(\mathbf{fe}') = 0$. From these assumptions we have

$$\mathbf{\Sigma} = \Lambda\Lambda' + \mathbf{\Psi} \tag{1}$$

where the matrix $\mathbf{\Sigma} = (\sigma_{ij})$ denotes variance-covariance matrix of $\mathbf{x}$ [3].

Albert [1] has given a theorem, that leads to a direct procedure for determining whether $\mathbf{\Sigma} - \mathbf{\Psi}$ is of rank k. This procedure does not verify that whether $\mathbf{\Sigma} - \mathbf{\Psi}$ positive definite. Suppose that the matrix $\mathbf{\Sigma}$ partitions as follows:

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} & \mathbf{\Sigma}_{13} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} & \mathbf{\Sigma}_{23} \\ \mathbf{\Sigma}_{31} & \mathbf{\Sigma}_{32} & \mathbf{\Sigma}_{33} \end{bmatrix}$$

Let k is the maxium rank of the submatrices of $\mathbf{\Sigma}$ that do not include diagonal elements and $\mathbf{\Sigma}_{11}, \mathbf{\Sigma}_{12} = \mathbf{\Sigma}'_{21}$ and $\mathbf{\Sigma}_{22}$ are square submatrices of order k and $\mathbf{\Sigma}_{12}$ is nonsingular. Then $\mathbf{\Sigma} - \mathbf{\Psi}$ is of rank k, if

$$\mathbf{\Sigma}_{12} = (\Sigma_{11} - \Psi_1)\, \mathbf{\Sigma}_{21}^{-1} (\Sigma_{22} - \Psi_2)$$

$$\mathbf{\Sigma}_{13} = (\Sigma_{11} - \Psi_1)\, \mathbf{\Sigma}_{21}^{-1} \mathbf{\Sigma}_{23}$$

$$\mathbf{\Sigma}_{32} = \mathbf{\Sigma}_{31} \mathbf{\Sigma}_{21}^{-1} (\Sigma_{22} - \Psi_2)$$

$$\mathbf{\Sigma}_{33} - \mathbf{\Psi}_3 = \mathbf{\Sigma}_{31} \mathbf{\Sigma}_{21}^{-1} \mathbf{\Sigma}_{23}.$$

Albert [2] has further shown that if $\mathbf{\Sigma}_{31}$ and $\mathbf{\Sigma}_{32}$ are also of rank k, then there is a uniquely determined $\mathbf{\Psi}$ such that $\mathbf{\Sigma} - \mathbf{\Psi}$ is of rank k.

Anderson and Rubin [5] gaved Teorem 5.1 that is a sufficient condition for identification of $\mathbf{\Psi}$ and $\mathbf{\Lambda}$ up to multiplication on the right by an orthogonal matrix is that if any row $\mathbf{\Lambda}$ is deleted there remain two disjoint submatrices of rank k.

Ihara and Kano[3] presumed that the matrix $\mathbf{\Lambda}$ in (1) satisfies the condition for identification of $\mathbf{\Psi}$ in Theorem 5.1 in [5]. Then, partitioning the matrices $\mathbf{\Sigma}$, $\mathbf{\Lambda}$ and $\mathbf{\Psi}$, they defined an estimator of Ψ_p by

$$\hat{\Psi}_p = s_{pp} - s'_{2p} S_{12}^{-1} s_{1p},$$

provided that the submatrix $\mathbf{S}_{12}$ is nonsingular. $\mathbf{S}$ is sample covariance matrix which is partitioned in the same fashion as $\mathbf{\Sigma}$.

Kano [6] proposed a non-iterative estimator using g-inverse matrix in factor analysis, which is a generalization of Ihara and Kano's estimator [3].

If there exists an explicit function $\mathbf{g}(\boldsymbol{\Sigma})$ of $\boldsymbol{\Sigma}$ such that $\boldsymbol{\Psi} = \mathbf{g}(\boldsymbol{\Sigma})$, then $\mathbf{g}(S)$ will be a good estimator of $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}$ can be easily estimated based on $\mathbf{S} - \mathbf{g}(\mathbf{S})$ [6].

Ihara and Kano [3] found such a function $\mathbf{g}$ and showed that the estimate $\hat{\boldsymbol{\Psi}} = \mathbf{g}(\mathbf{S})$ leads to a value rather close to maximum likelihood estimator (MLE) by using two real data sets.

Kano [6] partitions $\boldsymbol{\Lambda}$ as follow:

$$\Lambda = \begin{bmatrix} \lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{bmatrix} \begin{matrix} 1 \\ k_2 \\ k_3 \end{matrix}$$
$$k$$

where $\boldsymbol{\Sigma}$ and $\mathbf{S}$ are partitioned according to the above $\boldsymbol{\Lambda}$.

Since $\boldsymbol{\Lambda}_2$ and $\boldsymbol{\Lambda}_3$ are of full column rank under Anderson and Rubin's condition, there are a k_2 vector $\mathbf{a}_2$ and a k_3 vector $\mathbf{a}_3$ such as

$$\lambda_1 = \mathbf{a}_2'\boldsymbol{\Lambda}_2 = \mathbf{a}_3'\boldsymbol{\Lambda}_3. \tag{2}$$

Let $\mathbf{A}^-$ be any generalized inverse (g-inverse) matrix of $\mathbf{A}$. From the equation (2), Kano [6] led to the following relation:

$$\begin{aligned} \lambda_1 &= \mathbf{a}_2' = \mathbf{a}_3'\boldsymbol{\Lambda}_3\boldsymbol{\Lambda}_2'\mathbf{a}_2 \\ &= \mathbf{a}_3'\boldsymbol{\Lambda}_3\boldsymbol{\Lambda}_2'\left(\boldsymbol{\Lambda}_3\boldsymbol{\Lambda}_2'\right)^- \boldsymbol{\Lambda}_3\Lambda_2'\mathbf{a}_2 \\ &= \lambda_1\boldsymbol{\Lambda}_2'\left(\boldsymbol{\Lambda}_3\boldsymbol{\Lambda}_2'\right)\boldsymbol{\Lambda}_3\lambda_1' \\ &= \sigma_{12}\boldsymbol{\Sigma}_{32}'\sigma_{31}. \end{aligned}$$

Then from this equation ψ_1 is found in equation (3)

$$\psi_1 = \sigma_{11} - \sigma_{12}\boldsymbol{\Sigma}_{32}'\sigma_{31}. \tag{3}$$

2 Theoretical Aspects

In this paper by using generalized inverse matrix we give a theorem below, defining estimators of unique variances in factor analysis, which is generalization of Albert's Theorem[1,2].

Theorem 2.1 *Let $\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}' + \boldsymbol{\Psi}$ be covariance matrix of observable vector $\mathbf{x}$ and the matrices $\boldsymbol{\Sigma}$, $\boldsymbol{\Lambda}$ and $\boldsymbol{\Psi}$ partition as follows:*

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \boldsymbol{\Sigma}_{13} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \boldsymbol{\Sigma}_{23} \\ \boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32} & \boldsymbol{\Sigma}_{33} \end{bmatrix} \begin{matrix} m \\ n \\ t \end{matrix} \qquad \boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_1 \\ \boldsymbol{\Lambda}_2 \\ \Lambda_3 \end{bmatrix} \qquad \boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Psi}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Psi}_3 \end{bmatrix}.$$
$$m \quad n \quad t$$

Suppose that rank $\mathbf{\Sigma}_{12} = m$. *Then* $\mathbf{\Sigma} - \mathbf{\Psi}$ *is of rank* m *if*

$$\mathbf{\Sigma}_{21} = (\mathbf{\Sigma}_{22} - \mathbf{\Psi}_2)\,\mathbf{\Sigma}_{12}^{-}\,(\mathbf{\Sigma}_{11} - \mathbf{\Psi}_1)$$

$$\mathbf{\Sigma}_{31} = \mathbf{\Sigma}_{32}\mathbf{\Sigma}_{12}^{-}\,(\mathbf{\Sigma}_{11} - \mathbf{\Psi}_1)$$

$$\mathbf{\Sigma}_{23} = (\mathbf{\Sigma}_{22} - \mathbf{\Psi}_2)\,\mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{13}$$

$$\mathbf{\Sigma}_{33} - \mathbf{\Psi}_3 = \mathbf{\Sigma}_{32}\mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{13}$$

$$(\mathbf{\Sigma}_{22} - \mathbf{\Psi}_2)\left(\mathbf{I} - \mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{12}\right) = \mathbf{0}_{nxn}$$

$$\mathbf{\Sigma}_{32}\left(\mathbf{I} - \mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{12}\right) = \mathbf{0}_{txn}.$$

Furthermore, if $m = n$, $\mathbf{\Sigma}_{13}$ *and* $\mathbf{\Sigma}_{23}$ *are also full row rank, then there is a uniquely determined* $\mathbf{\Psi}$ *such that* $\mathbf{\Sigma} - \mathbf{\Psi}$ *is of rank* m.

Proof 2.1 *Premultiplication of* $\mathbf{\Sigma} - \mathbf{\Psi}$ *by*

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_{mxm} & \mathbf{0}_{mxn} & \mathbf{0}_{mxt} \\ -(\mathbf{\Sigma}_{22} - \mathbf{\Psi}_2)_{nxn}\mathbf{\Sigma}_{12_{nxm}}^{-} & \mathbf{I}_{nxn} & \mathbf{0}_{nxt} \\ -\mathbf{\Sigma}_{32_{txn}}\mathbf{\Sigma}_{12_{nxm}}^{-} & \mathbf{0}_{txn} & \mathbf{I}_{txt} \end{bmatrix}_{pxp}$$

and post-multiplication by

$$\mathbf{\Theta} = \begin{bmatrix} \mathbf{I}_{mxm} & \mathbf{0}_{mxn} & \mathbf{0}_{mxt} \\ -\mathbf{\Sigma}_{12_{nxm}}^{-}(\mathbf{\Sigma}_{11} - \mathbf{\Psi}_1)_{mxm} & \mathbf{I}_{nxn} & -\mathbf{\Sigma}_{12_{nxm}}^{-}\mathbf{\Sigma}_{13_{mxt}} \\ \mathbf{0}_{txm} & \mathbf{0}_{txn} & \mathbf{I}_{txt} \end{bmatrix}_{pxp}$$

then we get,

$$P(\mathbf{\Sigma} - \mathbf{\Psi})\Theta = \begin{bmatrix} \mathbf{0} & \mathbf{\Sigma}_{12} & \mathbf{0} \\ A & B & C \\ D & E & F \end{bmatrix}$$

where

$$A = -(\mathbf{\Sigma}_{22} - \mathbf{\Psi}_2)\,\mathbf{\Sigma}_{12}^{-}\,(\Sigma_{11} - \Psi_1) + \Sigma_{21},$$

$$B = -(\mathbf{\Sigma}_{22} - \mathbf{\Psi}_2)\left(\mathbf{I} - \mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{12}\right),$$

$$C = -(\mathbf{\Sigma}_{22} - \mathbf{\Psi}_2)\,\mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{13} + \mathbf{\Sigma}_{23},$$

$$D = -\mathbf{\Sigma}_{32}\mathbf{\Sigma}_{12}^{-}\,(\mathbf{\Sigma}_{11} - \mathbf{\Psi}_1) + \mathbf{\Sigma}_{31},$$

$$E = -\mathbf{\Sigma}_{32}\mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{12} + \mathbf{\Sigma}_{32},$$

$$F = -\mathbf{\Sigma}_{32}\mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{13} + \mathbf{\Sigma}_{33} - \mathbf{\Psi}_3.$$

Since the matrices $\mathbf{P}$ *and* $\mathbf{\Theta}$ *are nonsingular, then*

$$rank\,(\mathbf{P}\,(\mathbf{\Sigma} - \mathbf{\Psi})\,\mathbf{\Theta}) = rank\,(\mathbf{\Sigma} - \mathbf{\Psi})\,.$$

So, the rank of matrices $\mathbf{P}(\mathbf{\Sigma}-\mathbf{\Psi})\mathbf{\Theta}$ *and* $(\mathbf{\Sigma}-\mathbf{\Psi})$ *are equal to m if*

$$\mathbf{\Sigma}_{21}=(\mathbf{\Sigma}_{22}-\mathbf{\Psi}_2)\mathbf{\Sigma}_{12}^{-}(\mathbf{\Sigma}_{11}-\mathbf{\Psi}_1) \tag{4}$$

$$\mathbf{\Sigma}_{31}=\mathbf{\Sigma}_{32}\mathbf{\Sigma}_{12}^{-}(\mathbf{\Sigma}_{11}-\mathbf{\Psi}_1) \tag{5}$$

$$\mathbf{\Sigma}_{23}=(\mathbf{\Sigma}_{22}-\mathbf{\Psi}_2)\mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{13} \tag{6}$$

$$\mathbf{\Sigma}_{33}-\mathbf{\Psi}_3=\mathbf{\Sigma}_{32}\mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{13} \tag{7}$$

$$(\mathbf{\Sigma}_{22}-\mathbf{\Psi}_2)\left(\mathbf{I}-\mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{12}\right)=\mathbf{0}_{nxn} \tag{8}$$

$$\mathbf{\Sigma}_{32}\left(\mathbf{I}-\mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{12}\right)=\mathbf{0}_{txn}. \tag{9}$$

Since $\mathbf{\Sigma}_{12}$ *and* Σ_{23} *are full row rank, pre-multiplication of the equation (5) by* $\mathbf{\Sigma}_{32}^{-}$ *and* Σ_{12}*, respectively, then gives,*

$$\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{32}^{-}\mathbf{\Sigma}_{31}=\mathbf{\Sigma}_{11}-\mathbf{\Psi}_1.$$

From this equation we have

$$\mathbf{\Psi}_1=\mathbf{\Sigma}_{11}-\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{32}^{-}\mathbf{\Sigma}_{31} \tag{10}$$

Post multiplication of the equation (6) by $\mathbf{\Sigma}_{13}^{-}$ *and* $\mathbf{\Sigma}_{12}$*, respectively, since* $\mathbf{\Sigma}_{13}$ *is full row rank, then we get*

$$\mathbf{\Sigma}_{23}\mathbf{\Sigma}_{13}^{-}\mathbf{\Sigma}_{12}=(\mathbf{\Sigma}_{22}-\mathbf{\Psi}_2)\mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{12}. \tag{11}$$

From the equation (8), the right hand side of the equation (11) will be equal to $(\mathbf{\Sigma}_{22}-\mathbf{\Psi}_2)$*, so we have*

$$\begin{aligned}\mathbf{\Sigma}_{23}\mathbf{\Sigma}_{13}^{-}\mathbf{\Sigma}_{12} &= \mathbf{\Sigma}_{22}-\mathbf{\Psi}_2\\ \mathbf{\Psi}_2 &= \mathbf{\Sigma}_{22}-\mathbf{\Sigma}_{23}\mathbf{\Sigma}_{13}^{-}\mathbf{\Sigma}_{12}.\end{aligned} \tag{12}$$

From the equation (7), we have

$$\mathbf{\Psi}_3=\mathbf{\Sigma}_{33}-\mathbf{\Sigma}_{32}\mathbf{\Sigma}_{12}^{-}\mathbf{\Sigma}_{13}. \tag{13}$$

So $\mathbf{\Psi}$ *can be uniquely determined and the proof is completed.*

Estimation for $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$ and $\mathbf{\Psi}_3$ are obtained if the corresponding sample covariance matrices are used in (10), (12) and (13) instead of the population covariance matrices we find:

$$\hat{\mathbf{\Psi}}_1=\mathbf{S}_{11}-\mathbf{S}_{12}\mathbf{S}_{32}^{-}\mathbf{S}_{31} \tag{14}$$

$$\hat{\mathbf{\Psi}}_2=\mathbf{S}_{22}-\mathbf{S}_{23}\mathbf{S}_{13}^{-}\mathbf{S}_{12} \tag{15}$$

$$\hat{\mathbf{\Psi}}_3=\mathbf{S}_{33}-\mathbf{S}_{32}\mathbf{S}_{12}^{-}\mathbf{S}_{13}. \tag{16}$$

As a result writing $m=1$ for equation (10) gives us the equation (3). $m=n=t=k$ for equations (10), (12) and (13) gives us the Albert's Theorem [2].

References

[1] A.A. Albert, The matrices of factor analysis, *Proc.Nat.Acad.Sci.,* **30** (1944a), 90-95.

[2] A.A. Albert, The minimum rank of a correlation matrix, *Proc.Nat.Acad.Sci.,* **30** (1944b), 144-146.

[3] M. Ihara, Y. Kano, A new estimator of the uniqueness in factor analysis, *Psychometrika,* **51** (1986), 563 - 566.

[4] R.A. Johnson, D.W. Wichern, *Applied Multivariate Statistical Analysis,* Wiley, United States of America, 2002.

[5] T.W. Anderson, H. Rubin, Statistical inference in factor analysis, *Proc.3rd Berkeley Symp,* **5** (1956), 111 - 150.

[6] Y. Kano, A new estimation procedure using g-inverse matrix in factor analysis, *Math. Japonica,* **34:1** (1989), 43-52.

Received: October 29, 2007