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Darboux Function in a Hypersurface of a Riemannian Manifold with Semi-Symmetric Metric Connection

Füsun Özen [1], **Sezgin Altay**

Istanbul Technical University
Department of Mathematics, Istanbul, Turkey

S. Aynur Uysal

Dogus University, Department of Mathematics
Istanbul, Turkey

Abstract

In 1970, Yano, [1], studied Riemannian manifolds which admit semi-symmetric metric connections whose curvature tensors vanish (see also [2]). The properties of a Riemannian manifold admitting a semi-symmetric metric connection were studied by many authors ([1], [3]). In [3], an expression of the curvature tensor of a manifold was obtained under assumption that the manifold admits a semi-symmetric metric connection with vanishing curvature tensor and recurrent torsion tensor.

In this paper, we study a Darboux function in hypersurface of a Riemannian manifold with semi-symmetric metric connection. The purpose of this paper is that the relations between the Darboux function with respect to the linear connection and the Darboux function with respect to the Levi-Civita connection of a Riemannian manifold are obtained. In this paper, some theorems about this function are proved.

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[1] fozen@itu.edu.tr

1 Introduction

Let M be an n-dimensional Riemannian manifold of class C^∞. Let ∇^* be a linear connection on M. Then the torsion tensor T of ∇^* is given by

$$T(X,Y) = \nabla^*_X Y - \nabla^*_Y X - [X,Y] \tag{1.1}$$

for any vector fields X and Y in M and is of type (1,2).

When the torsion tensor T satisfies the relation

$$T(X,Y) = w(Y)X - w(X)Y \tag{1.2}$$

for a 1-form w, the connection ∇^* is said to be semi-symmetric.

We assume that g is given as a Riemannian metric and ∇^* satisfies the condition

$$\nabla^* g = 0 \tag{1.3}$$

Such a linear connection is called a metric connection. The equation (1.3) means

$$\nabla^*_X(g(Y,Z)) = g(\nabla^*_X Y, Z) + g(Y, \nabla^*_X Z)$$

for any vector fields X, Y and Z.

If we denote by ∇ the Levi-Civita connection with respect to the Riemannian metric g, we have

$$\nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for any vector fields X, Y and Z.

Particularly, a semi-symmetric metric connection of M_n is given by

$$\nabla^*_X Y = \nabla_X Y + w(Y)X - g(X,Y)\rho \tag{1.4}$$

where ρ is a vector field defined by

$$g(X,\rho) = w(X) \tag{1.5}$$

Such a linear connection ∇^* is called a semi-symmetric metric connection, [5]. This linear connection has also appeared in [5], [6], [7] and [8].

We denote the curvature tensor of this connection by

$$R^*(X,Y)Z = R(X,Y)Z - P(Y,Z)X + P(X,Z)Y - g(Y,Z)AX + g(X,Z)AY \tag{1.6}$$

where the relation

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is the curvature tensor of the connection of a Riemannian manifold M with respect to the Riemannian connection ∇ and P is a tensor field of type $(0,2)$ defined on M by

$$P(X,Y) = (\nabla_X w)(Y) - w(X)w(Y) + \frac{1}{2}w(\rho)g(X,Y)$$

and A is a tensor field of type (1,1) defined by

$$g(AX,Y) = P(X,Y)$$

for any vector fields X and Y, [1] and [9].

J.A Shouten,[3], called a linear connection Γ^h_{ji} whose torsion tensor ${S_{ji}}^h$ is of the form

$${S_{ji}}^h = \Gamma^h_{ji} - \Gamma^h_{ij} = \delta^h_j w_i - \delta^h_i w_j \tag{1.7}$$

where w_i is a certain 1-form of a semi-symmetric connection. Since ${S_{ji}}^h$ is given by (1.7), using the relation ${S^h}_{ji} = {S_{bj}}^a g^{bh} g_{ai}$ is given by

$${S^h}_{ji} = w_j \delta^h_i - g_{ij} w^h$$

where $w^h = g^{hi} w_i$. Thus, the components Γ^h_{ji} of the semi-symmetric metric connection ∇^* are in the form

$$\Gamma^h_{ji} = \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} + \delta^h_j w_i - g_{ij} w^h \tag{1.8}$$

Denoting by ${R_{kji}}^{h*}$ the curvature tensor of the connection Γ^h_{ji} and by ${R_{kji}}^h$ that of $\left\{ \begin{smallmatrix} h \\ ji \end{smallmatrix} \right\}$, we obtain

$${R_{kji}}^{h*} = {R_{kji}}^h - \delta^h_k P_{ji} + \delta^h_j P_{ki} - {P_k}^h g_{ji} + {P_j}^h g_{ki} \tag{1.9}$$

where $P_{ij} = \nabla_j w_i - w_i w_j + \frac{1}{2} w_h w^h g_{ij}$ and ${P_k}^h = P_{km} g^{mh}$ and ∇ denotes the covariant differentiation with respect to the Christoffel symbols. If we consider the covariant derivative in the formulas with respect to Eisenhart (not Yano) and we use (1.3), we get

$$\Gamma^i_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \Omega^i_{jk} \quad , \qquad {S_{jk}}^i = \delta^i_k w_j - \delta^i_j w_k \tag{1.10}$$

where

$$\Gamma^i_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \delta^i_k w_j - g_{jk} w^i \ , \qquad \Omega^i_{jk} = \frac{1}{2}({S_{jk}}^i + g^{im}(g_{hk} {S_{jm}}^h + g_{hj} {S_{km}}^h))$$

Specially, let us consider the torsion tensor of the connection ∇^* is given by (1.10) where w is a 1-form and $w_i = g_{ij} w^j$.

The covariant derivatives of the contravariant vector A^h relative to ∇^* and ∇ denoted, respectively, by $\nabla^* A$ and ∇A are related by

$$\nabla^*_k A^h = \nabla_k A^h + \Omega^h_{mk} A^m \tag{1.11}$$

where $\nabla_k A^h = \partial_k A + \left\{ {h \atop mk} \right\} A^m$.

Let $M_n(\nabla, g)$ be a hypersurface with coordinates $x^i (i = 1, 2, ..., n)$ of Riemannian manifold $M_{n+1}(\nabla, g)$ with coordinates $y^\alpha (\alpha = 1, 2, ..., n+1)$. Suppose that the metrics of M_n and M_{n+1} are positive definite and that they are given, respectively, by $g_{ij} dx^i dx^j$ and $g_{\alpha\beta} dy^\alpha dy^\beta$ which are connected by the relation

$$g_{ij} = g_{\alpha\beta} y^\alpha_i y^\beta_j \quad (i, j = 1, 2, ..., n; a, b = 1, 2, ..., n+1) \ , \qquad y^\alpha_i = \frac{\partial y^\alpha}{\partial x^i} \tag{1.12}$$

where y^α_i denotes the covariant derivative of y^α with respect to x^i.

If n^α are the components of a unit vector in M_{n+1} normal to M_n, these satisfy the relations

$$g_{\alpha\beta} n^\alpha n^\beta = 1 \tag{1.13}$$

Let $\underset{r}{v}^i$ $(r = 1, 2, ..., n)$ be the contravariant components of the n independent vector fields $\underset{r}{\vec{v}}$ of an orthogonal ennuple in M_n which satisfy the condition

$$g_{ij} \underset{r}{v}^i \underset{p}{v}^j = \delta^r_p \qquad (r, p = 1, 2, ..., n) \tag{1.14}$$

Suppose that $\underset{r}{v}^a$ be the contravariant components of the net consider in M_n relative to M_{n+1}, then we have, [10],

$$\underset{r}{v}^a = y^a_i \underset{r}{v}^i \qquad g_{ab} \underset{r}{v}^a \underset{p}{v}^b = \delta^r_p, \qquad g_{ab} n^a \underset{r}{v}^b = 0 \tag{1.15}$$

On the other hand, for the tensorel derivative of y^α_i, we have, [10]

$$\dot{\nabla}_j y^\alpha_i = \frac{\partial^2 y^\alpha}{\partial x^j \partial x^i} + \left\{ {\alpha \atop \beta\gamma} \right\} y^\beta_i y^\gamma_j - \left\{ {m \atop ij} \right\} y^\alpha_m \tag{1.16}$$

and

$$\dot{\nabla}_j y^\alpha_i = \Omega_{ij} n^\alpha \tag{1.17}$$

where $\dot{\nabla}$ is the tensorel derivative with respect to ∇.

Similarly, the tensorel derivative of the contravariant vector n^a is defined by

$$\dot{\nabla}_k n^\alpha = -\Omega_{km} g^{mj} y^\alpha_j \tag{1.18}$$

Suppose that a Riemannian manifold M with metric g_{ab} admits a semi-symmetric metric connection with connection coefficient Γ^a_{bc} given by

$$\Gamma^a_{bc} = \left\{ {a \atop bc} \right\} + \delta^a_c w_b - g_{bc} w^a \tag{1.19}$$

where $\left\{ {a \atop bc} \right\}$ are the Christoffel symbols of M, w_a is a covector field and $w^a = g^{ab}w_b$.

Let M_n be a submanifold of M_m with induced metric g_{ij} and induced Christoffel functions $\left\{ {i \atop jk} \right\}$. If we take $y^c_h = \frac{\partial y^c}{\partial x^h}$ and $y^h_a = g^{hi}g_{ab}y^b_i$ then we have $w_h = y^c_h w_c$ and $w^h = g^{hm}w_m$. Suppose that $c^a_1, c^a_2, ..., c^a_{m-n}$ are unit orthogonal normal fields on M. Decomposing w^a into its unique tangential and normal components along M, we get

$$w^a = w^h y^a_h + \alpha^x c^a_x \tag{1.20}$$

where the summation in the index x runs over the range $x = 1, 2, ..., m-n$.

On the other hand, by [11], we obtain the following relations

$$\frac{\delta}{\delta s}\underset{r}{v}^i = \underset{r}{v}^h \dot{\nabla}_h \underset{r}{v}^i = \overset{p}{\underset{r}{\zeta}}\underset{p}{v}^i, \qquad (r \neq p,\ r, p = 1, 2, ..., n) \tag{1.21}$$

$$\frac{\delta}{\delta s}\underset{r}{v}^a = \underset{r}{v}^h \dot{\nabla}_h \underset{r}{v}^a = \underset{rr}{\kappa} n^a + \overset{p}{\underset{r}{\zeta}}\underset{p}{v}^a \tag{1.22}$$

The tensorel derivative formulae for the vector field $\underset{r}{\vec{v}}$, we find

$$\underset{r}{v}^h \dot{\nabla}^*_h \underset{p}{v}^i = \underset{r}{v}^h \dot{\nabla}_h \underset{p}{v}^i + \Omega^i_{kh}\underset{p}{v}^k \underset{r}{v}^h \tag{1.23}$$

$$\underset{r}{v}^h \dot{\nabla}_h \underset{r}{v}^m = \kappa_g b^m \tag{1.24}$$

2 A Darboux function in a Riemannian manifold with semi-symmetric metric connection

Let us consider a manifold with semi-symmetric metric connection such that the torsion tensor of this manifold satisfies the condition (1.10). In this case, the linear connection of $M(\nabla^*, g)$ satisfies the following conditions

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + \Omega^i_{jk} \tag{2.1}$$

where

$$\Omega^i_{jk} = \frac{1}{2}({S_{jk}}^i + g^{im}(g_{hk}{S_{jm}}^h + g_{hj}{S_{km}}^h)) \ , \quad {S_{ji}}^h = \delta^h_i w_j - \delta^h_j w_i$$

By using the above equation, the tensorel derivatives of y^α_i relative to ∇^* and ∇ are denoted by $\dot{\nabla}^*_k y^\alpha_i$ and $\dot{\nabla}_k y^a_i$, respectively, and are defined

$$\dot{\nabla}^*_k y^\alpha_i = \Omega^*_{ik} n^\alpha \tag{2.2}$$

The coefficients Ω^*_{ik} are the components of a symmetric covariant tensor of the second order in the x's is obvious from the fact that the functions $\dot{\nabla}^*_k y^\alpha_i$ are of this nature. From (1.13) and (2.2), we have

$$\Omega^*_{ik} = g_{\alpha\beta}(\dot{\nabla}^*_k y^\alpha_i) n^\beta \tag{2.3}$$

It is easy to see that the tensorel derivative of A, relative to $M_n(\nabla^*, g)$ and $M_{n+1}(\nabla^*, g)$ are related by

$$\dot{\nabla}^*_k A = y^c_k \dot{\nabla}^*_c A \tag{2.4}$$

If we take the tensorel derivative of (1.13) relative to $\dot{\nabla}^*$, we get $g_{\alpha\beta} n^\alpha \dot{\nabla}^*_k n^\beta = 0$, so that $\dot{\nabla}^*_k n^\beta$ regarded as a vector in $M_n(\nabla^*, g)$ is orthogonal to n^α. Let us express it as

$$\dot{\nabla}^*_k n^\alpha = -\Omega^*_{km} g^{mj} y^\alpha_j \tag{2.5}$$

the expression $\dot{\nabla}^*_k n^\alpha = \dot{\nabla}_k n^\alpha + \Omega^\alpha_{\beta\gamma} n^\beta y^\gamma_k$ can be put in the form

$$\dot{\nabla}^*_k n^\alpha = \dot{\nabla}_k n^\alpha + \Omega^\alpha_{\beta\gamma} n^\beta y^\gamma_k = \dot{\nabla}_k n^\alpha + w_\beta n^\beta y^\alpha_k \tag{2.6}$$

Let $\underset{r}{v}^i$ $(r = 1, 2, ..., n)$ be the contravariant components of the n independent vector fields $\underset{r}{\vec{v}}$ in $M_n(\nabla^*, g)$ which satisfy the condition

$$g_{ij} \underset{r}{v}^i \underset{r}{v}^j = 1 \tag{2.7}$$

Let $\underset{r}{v}^a$ and $\underset{r}{v}^i$ be, respectively, the contravariant components of $\underset{r}{\vec{v}}$ relative to $M_{n+1}(\nabla^*, g)$ and $M_n(\nabla^*, g)$, we have

$$\underset{r}{v}^\alpha = y^\alpha_i \underset{r}{v}^i \ , \qquad g_{\alpha\beta} n^\alpha \underset{r}{v}^\beta = 0 \tag{2.8}$$

On the other hand, the functions

$$\underset{rp}{\kappa}^* = \Omega^*_{ik} \underset{r}{v}^i \underset{p}{v}^k \tag{2.9}$$

may be regarded as the invariants of the geodesic torsion of the curve C belonging to the congruence with tangent vector $\underset{r}{\vec{v}}$.

In particular, taking $r = p$ in (2.9), we get

$$\underset{rr}{\kappa}^* = \Omega^*_{ik} \underset{r}{v}^i \underset{r}{v}^k \tag{2.10}$$

which we call it the normal curvature of $M_n(\nabla^*, g)$ in the direction of the vector of components $\underset{r}{v}^k$.

On the other hand, using (2.6), (2.9) and (2.10), we have

$$g_{\alpha\beta}(\underset{p}{v}^k\dot{\nabla}^*_k n^\alpha)\underset{r}{v}^\beta = -\underset{rp}{\kappa}^* \ , \qquad g_{\alpha\beta}(\underset{r}{v}^k\dot{\nabla}^*_k n^\alpha)\underset{r}{v}^\beta = -\underset{rr}{\kappa}^* \tag{2.11}$$

In [12], the author introduced the operator ∇^* which enabled us to generalised the Darboux function of an ordinary space to the case of a hypersurface in a Riemannian manifold. We obtain an expression for this function in terms of various curvatures of a congruence in $M_{n+1}(\nabla^*, g)$ and a curve in $M_n(\nabla^*, g)$.

Let $C : x^i : x^i(s)$ be any curve in $M_n(\nabla^*, g)$ passing through a point P and $\underset{r}{v}^i$ and $\underset{r}{v}^a$ $(g_{ij}\underset{r}{v}^i\underset{p}{v}^j = \delta^r_p)$, the contravariant components of the unit tangent vector to the curve in $M_n(\nabla^*, g)$ and $M_{n+1}(\nabla^*, g)$, respectively.

Let the operators D^* and D be defined in the following form

$$D^* = y^a_i g^{ij}\dot{\nabla}^*_j \qquad D = y^a_i g^{ij}\dot{\nabla}_j \tag{2.12}$$

Consider the expressions $\underset{r}{\vec v}\frac{\dot\delta D^*\vec n}{\delta s_r}\underset{p}{\vec v}$ and $\underset{r}{\vec v}\frac{\delta D\vec n}{\delta s_r}\underset{p}{\vec v}$ where s_p is the arc length along a curve $c_p(p = 1, 2, ..., n)$ of the orthogonal ennuple in $M^*_n(\nabla^*, g)$. By means of (2.12), we find

$$\begin{aligned}\frac{\dot\delta D^*\vec n}{\delta s_r} &= \underset{r}{v}^h\dot{\nabla}^*_h(y^a_i g^{ik}\dot{\nabla}^*_j n^c)\\ &= \underset{r}{v}^h(\dot{\nabla}^*_h y^a_i)g^{ik}\dot{\nabla}^*_j n^c + \underset{r}{v}^h y^a_i(\dot{\nabla}^*_h g^{ik})\dot{\nabla}^*_j n^c + \underset{r}{v}^h y^a_i g^{ik}\dot{\nabla}^*_h\dot{\nabla}^*_j n^c\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\frac{\delta D\vec n}{\delta s_r} &= \underset{r}{v}^h\dot{\nabla}_h(y^a_i g^{ik}\dot{\nabla}_j n^c)\\ &= \underset{r}{v}^h(\dot{\nabla}_h y^a_i)g^{ik}(\dot{\nabla}_j n^c) + \underset{r}{v}^h y^a_i(\dot{\nabla}_h g^{ik})(\dot{\nabla}_j n^c) + \underset{r}{v}^h y^a_i g^{ik}\dot{\nabla}_h\dot{\nabla}_j n^c\end{aligned} \tag{2.14}$$

In a Riemannian hypersurface with semi-symmetric metric connection, for the extended Darboux function relative to the congruence (λ),[13], and $\vec\lambda = \vec n$, after some calculations, we get

$$\mathcal{D}^*_{rrp} = \mathcal{D}_{rrp} + \underset{rr}{\kappa}w_j\underset{p}{v}^j + \underset{rp}{\kappa}w_j\underset{r}{v}^j - \Omega_{mn}w^m\underset{p}{v}^n g_{jk}\underset{r}{v}^j\underset{r}{v}^k \ , \qquad r \neq p \tag{2.15}$$

$$\mathcal{D}^*_{rpr} = \mathcal{D}_{rpr} + 2\underset{rp}{\kappa}w_j\underset{r}{v}^j + (\underset{p}{v}^k\dot{\nabla}_k\alpha)g_{nj}\underset{r}{v}^n\underset{r}{v}^j, \qquad r \neq p \tag{2.16}$$

If α, in (1.20), is absolute constant then the equation (2.16) reduces to

$$\mathcal{D}^*_{rpr} = \mathcal{D}_{rpr} + 2\underset{rp}{\kappa}w_j\underset{r}{v}^j \ , \quad r \neq p \tag{2.17}$$

and

$$\mathcal{D}^*_{rpp} = \mathcal{D}_{rpp} + \underset{pp}{\kappa}w_j\underset{r}{v}^j + \underset{rp}{\kappa}w_j\underset{p}{v}^j - \Omega_{js}w^s\underset{r}{v}^j g_{kn}\underset{p}{v}^k\underset{p}{v}^n \ , \qquad r \neq p \tag{2.18}$$

where $\mathcal{D}^*_{rrp}$ and $\mathcal{D}_{rrp}$, respectively, Darboux functions of the direction $\underset{r}{\vec{v}}$ with respect to the direction $\underset{p}{\vec{v}}$, $(r \neq p)$ relative to the congruence $\vec{n}$ for the linear connection and the Levi-Civita connection.

In the following two theorems, let the torsion S of the connection ∇^* be recurrent with respect to connection ∇^*, i.e, the condition

$$(\nabla^*_X S)(Y,Z) = \lambda(X) S(Y,Z) \tag{2.19}$$

holds, where λ is a 1-form. Using (1.15), (1.20), (2.1), (2.2) and (2.5), after some calculations, we obtain

$$g_{cd}\underset{p}{v}^d \underset{p}{v}^\gamma n^\beta \underset{r}{v}^k \dot{\nabla}^*_k S_{\beta\gamma}{}^c = \Omega^*_{ks}\underset{r}{v}^k w^s + \underset{r}{v}^k \dot{\nabla}_k \alpha \tag{2.20}$$

and from (2.6), we get

$$g_{cd}\underset{p}{v}^d \underset{p}{v}^\gamma n^\beta \underset{r}{v}^k \dot{\nabla}^*_k S_{\beta\gamma}{}^c = \Omega_{ks}\underset{r}{v}^k w^s + \underset{r}{v}^k \dot{\nabla}_k \alpha - \alpha w_h \underset{r}{v}^h \tag{2.21}$$

From (2.19) and (2.21), we get

$$\Omega_{ks}\underset{r}{v}^k w^s = (\theta + \eta)\alpha - \underset{r}{v}^k \dot{\nabla}_k \alpha \tag{2.22}$$

where

$$\theta = w_k \underset{r}{v}^k \ , \ \eta = \lambda_k \underset{r}{v}^k \tag{2.23}$$

We consider the Darboux function belonging to $M_n(\nabla^*, g)$ and its associate Riemannian manifold (M_n, g). Then, we prove the following theorems:

THEOREM 2.1 *Let the directions of the Darboux function $\mathcal{D}_{rpp}$ belonging to Riemannian manifold $M_n(\nabla, g)$ be tangent directions of the lines of curvature belonging to this manifold. If the torsion tensor S of the connection ∇^* satisfies the condition (2.19) and*

$$\underset{r}{v}^k \partial_k ln\alpha^2 = 2(\theta + \eta) \tag{2.24}$$

then

$$\mathcal{D}^*_{rpp} = D_{rpp}$$

Proof. According to [10], the directions of $\underset{r}{\vec{v}}$ in M_n determined by the symmetric covariant tensor Ω_{ks} are those which satisfy

$$(\Omega_{ks} - \underset{rr}{\kappa} g_{ks})\underset{r}{v}^k = 0 \quad (r = 1, 2, ..., n) \tag{2.25}$$

From (2.22) and (2.24), we have

$$\Omega_{ks}\underset{r}{v}^k w^s = 0 \tag{2.26}$$

With the help of (2.25) and (2.26), we get

$$\underset{rr}{\kappa} = 0 \qquad or \qquad \vec{w}\ is\ orthogonal\ to\ \underset{r}{\vec{v}} \tag{2.27}$$

Since the tangent directions of lines of curvature are not the tangent directions of asymptotic curves, $\vec{w}$ is orthogonal to $\underset{r}{\vec{v}}$.

Using (2.18), (2.25)-(2.27), the proof is completed

THEOREM 2.2 *Let $M_n(\nabla, g)$ be a totally umbilical hypersurface ($M \neq 0$). Then, the Darboux functions $\mathcal{D}^*_{rrp}$ and $\mathcal{D}^*_{rpp}$ of $M_n(\nabla^*, g)$ and the Darboux functions $\mathcal{D}_{rrp}$ and $\mathcal{D}_{rpp}$ of $M_n(\nabla, g)$ are equal.*

Proof. Since $M_n(\nabla, g)$ is totally umbilical hypersurface, we have

$$\Omega_{ks} = \frac{M}{n} g_{ks} \tag{2.28}$$

On the other hand, from the expressions (2.15), (2.18) and (2.28), the proof is clear.

In [14], the authors find an explicit formula for the curvature tensor of the Levi-Civita connection ∇ in the case when the curvature tensor R^* of the metric connection ∇^* vanishes identically, i.e.

$$R^*(X, Y)Z = 0 \tag{2.29}$$

and its torsion fulfils additionally the condition

$$R^*.S = 0 \tag{2.30}$$

THEOREM 2.3 *Let the directions of the Darboux functions $\mathcal{D}_{rpr}$ belonging to a Riemannian manifold be tangent directions of the lines of curvature belonging to this manifold. If the torsion tensor S of the connection ∇^* satisfies the condition (2.19) and*

$$\underset{p}{v}^k \partial_k ln\alpha^2 = 2(\theta^* + \eta^*)\ , \qquad \theta^* = w_k \underset{p}{v}^k\ , \qquad \eta^* = \lambda_k \underset{p}{v}^k$$

then

$$\mathcal{D}^*_{rrp} = \mathcal{D}_{rrp}.$$

Proof. Using (1.15), (1.18), (1.20), (2.5) and (2.19), we get

$$\Omega_{ks} \underset{p}{v}^k w^s = (\theta^* + \eta^*)\alpha - \underset{p}{v}^k \dot{\nabla}_k \alpha$$

where $\theta^* = w_k \underset{p}{v}^k$, $\eta^* = \lambda_k \underset{p}{v}^k$. Using (2.15) and

$$(\Omega_{ks} - \underset{pp}{\kappa} g_{ks}) \underset{p}{v}^k = 0 \quad (p = 1, 2, ..., n)$$

and following the similar procedure in Theorem 2.1, the proof is completed.

It is shown that if a Riemannian manifold admits a semi-symmetric metric connection with ω as its associated 1-form is closed and recurrent torsion tensor, then the manifold admits a torse-forming vector field, [15]. Thus, we can say that this vector field is concircular.

It is known that,[16], if a conformally flat space admits a proper concircular vector field then the space is sub-projective space in the sense of Cartan. Thus we can state the following theorem:

THEOREM 2.4 *If a 1-form ω of a Riemannian manifold with semi-symmetric metric connection is closed and its torsion tensor satisfies the condition (2.19) and this manifold is conformally flat then the Darboux functions of this manifold are also Darboux functions of the sub-projective space.*

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