Some Fredholm Integration Operators
on a Hilbert Space of Holomorphic Functions
on the Unit Disc

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Abstract

In this paper, we study when $M_\phi$, $I_\phi$ or $J_\phi$ is a Fredholm operator on a Hilbert space which satisfies few natural axioms.

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§ 1. Introduction

Let $D$ be the open unit disc in the complex plane $\mathbb{C}$ and $H(D)$ be the set of all analytic functions on $D$. $H(\bar{D})$ denotes the set of all analytic functions on $\bar{D}$. In this paper, $\mathcal{H}$ is a Hilbert space in $H(D)$ which satisfies the following:

(1) $z\mathcal{H} \subset \mathcal{H}$.
(2) If $a \in D$ then $(z - a)\mathcal{H} \oplus \mathbb{C} = \mathcal{H}$.
(3) $\mathcal{H} \supseteq H(\bar{D})$.

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In this paper, we study the following three operators. If \( \phi \) is a function in \( H(D) \), put for \( z \in D \),

\[
(M_\phi f)(z) = \phi(z)f(z),
\]

\[
(I_\phi f)(z) = \int_0^z f'(\zeta)\phi(\zeta)d\zeta,
\]

\[
(J_\phi f)(z) = \int_0^z f(\zeta)\phi'(\zeta)d\zeta \quad (f \in H).
\]

Then \((M_\phi f)(z) = (I_\phi f)(z) + (J_\phi f)(z) + \phi(0)f(0)\). It is clear that \( I_\phi \) and \( J_\phi \) are never invertible.

Put \( M(H) = \{ \phi \in H(D) : M_\phi H \subseteq H \} \), \( I(H) = \{ \phi \in H(D) : I_\phi H \subseteq H \} \) and \( J(H) = \{ \phi \in H(D) : J_\phi H \subseteq H \} \). In this paper, we assume that \( H(\bar{D}) \subseteq M(H) \), \( z \in I(H) \) and \( z \in J(H) \).

§ 2. Multiplication operator \( M_\phi \)

When \( M(H) = H^\infty(D) \), A. Aleman [1] shows a more general result than Corollary 1 without the condition that \((z - a)H\) is dense.

Lemma 1. If \( p \) is a polynomial with no zeros on \( \partial D \) then \( \dim H/pH < \infty \).

Proof If \( |a| > 1 \) then \((z - a)^{-1} \in H(\bar{D})\) and so \((z - a)^{-1}\) belongs to \( M(H)\). Hence we may assume that the zeros of \( p \) are contained in \( D \). By hypothesis on \( H\), \( \dim H/(z - a)H = 1 \) and so \( \dim H/pH < \infty \).

Lemma 2. If \( M \) is a closed invariant subspace of \( M_z \) in \( H \) such that \( \dim H/M < \infty \), then there exists a polynomial \( p \) such that \( pH \subseteq M\).

Proof Let \( N = H \ominus M \) and \( S_z = P_N M_z|N \), then \( S_z \) is of finite rank because \( \dim N < \infty \). Hence there exists a polynomial \( p \) such that \( S_{p(z)} = p(S_z) = 0 \). Therefore \( pN \subseteq M \) and so \( pH \subseteq M \).

Theorem 1.

(1) If \( \phi = Bg \) where \( B \) is a finite Blaschke product, and both \( g \) and \( g^{-1} \) are in \( M(H) \) then \( M_\phi \) is a Fredholm operator.

(2) If \( M_\phi \) is a Fredholm operator on \( H \) then \( \phi = Bg \) when \( B \) is a finite Blaschke product, \( g \) is in \( M(H) \) and \( g^{-1} \) is in \( H \).

(3) For the \( g \) in (2), \( M_\phi \) is a Fredholm operator on \( H \) with index \( M_\phi \leq \).
index $M_g \leq 0$ and there exists a polynomial $q$ such that $q\mathcal{H} \subseteq g\mathcal{H}$ and the zeros are in $\mathbb{C}\setminus D$.

**Proof** (1) Suppose $\phi = Bg$, $B = \prod_{j=1}^{n}(z - a_j)/(1 - \overline{a_j}z)$, $\{a_j\} \subset D$, and both $g$ and $g^{-1}$ are in $\mathcal{M}(\mathcal{H})$. Since $\mathcal{M}(\mathcal{H}) \supseteq H(\bar{D})$, $\prod_{j=1}^{n}(1 - \overline{a_j}z)$ is invertible in $\mathcal{M}(\mathcal{H})$ and so $M_\phi(\mathcal{H}) = p\mathcal{H}$ where $p = \prod_{j=1}^{n}(z - a_j)$.

(2) If $M_\phi$ is a Fredholm operator then $\dim \mathcal{H}/M_\phi(\mathcal{H}) < \infty$ and so by Lemma 2 there exists a polynomial $p$ such that $\phi f = p$. Therefore $\phi$ can be factorized as $\phi = Bg$ where $B$ is a finite Blaschke product and $g \in \mathcal{H}$. For $\phi \in \mathcal{H}$ and $\prod_{j=1}^{n}(1 - \overline{a_j}z)\phi = \prod_{j=1}^{n}(z - a_j)g \in \mathcal{H}$ where $B = \prod_{j=1}^{n}(z - a_j)/(1 - \overline{a_j}z)$. Since $\ker \tau_{a_j} = (z - a_j)\mathcal{H}$, $g$ belongs to $\mathcal{H}$. By the similar argument, there exists a function $k$ in $\mathcal{H}$ and $gk = 1$ because $Bg f = p$. Thus $g^{-1}$ belongs to $\mathcal{H}$. We will prove that $g$ belongs to $\mathcal{M}(\mathcal{H})$. Since $B$ is a finite Blaschke product and $\ker \tau_a = (z-a)\mathcal{H}$ for $a \in D$, $\mathcal{H} = K + B\mathcal{H}$ where $K$ is a finite dimensional subspace such that each function in $K$ is a rational function whose poles are in $\mathbb{C}\setminus \bar{D}$. Since $g \in \mathcal{H}$ and $\mathcal{M}(\mathcal{H}) \supseteq H(\bar{D})$, $gK \subseteq \mathcal{H}$ and so $g\mathcal{H} \subseteq \mathcal{H}$ because $gB\mathcal{H} \subseteq \mathcal{H}$.

(3) By the proof of (2), $p\mathcal{H} \subseteq Bg\mathcal{H} \subseteq g\mathcal{H}$ and so the first statement is clear. Again by the proof of (2), the zeros of $p$ in $D$ is just the zeros of $B$. This implies that there exists a polynomial $q$ such that $q\mathcal{H} \subseteq g\mathcal{H}$ and $q$ does not have any zeros in $D$.

**Corollary 1.** Suppose that $(z - a)\mathcal{H}$ is dense in $\mathcal{H}$ whenever $a \in \partial D$. Then $M_\phi$ is a Fredholm operator on $\mathcal{H}$ if and only if $\phi = Bg$ where $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $\mathcal{M}(\mathcal{H})$.

§ 3. Integral operator $I_\phi$

It seems to have not been studied yet in this general setting as Theorem 2.

**Lemma 3.** If $\phi$ is a function in $\mathcal{I}(\mathcal{H})$ then $I_\phi(\mathcal{H}) = I_\phi(z\mathcal{H}) \subseteq z\mathcal{H}$. $I_\phi(\mathcal{H}) = z\mathcal{H}$ if and only if $\phi$ and $\phi^{-1}$ belongs to $\mathcal{I}(\mathcal{H})$.

**Proof** By the definition of $I_\phi$ the first statement is clear. We will show the second one. If both $\phi$ and $\phi^{-1}$ belong to $\mathcal{I}(\mathcal{H})$, then

$$z\mathcal{H} = I_1(\mathcal{H}) = I_\phi I_{\phi^{-1}}(\mathcal{H}) \subseteq I_\phi(z\mathcal{H}) \subseteq z\mathcal{H}$$
because $I_\phi$ and $I_{\phi^{-1}}$ are bounded on $\mathcal{H}$. Conversely if $I_\phi(\mathcal{H}) = z\mathcal{H}$ then there exists a function $g$ in $\mathcal{H}$ such that
\[
\int_0^z g'(\zeta)\phi(\zeta)d\zeta = z \quad \text{and so} \quad g'(z)\phi(z) = 1.
\]
Hence $\phi^{-1} \in H(D)$ and
\[
z\mathcal{H} = I_1(\mathcal{H}) = I_{\phi^{-1}}I_\phi(\mathcal{H}) = I_{\phi^{-1}}(z\mathcal{H})
\]
and so both $\phi$ and $\phi^{-1}$ belong to $\mathcal{I}(\mathcal{H})$.

**Lemma 4.** If $p$ is a polynomial then $I_p(\mathcal{H}) + \mathbb{C} \supset p^2\mathcal{H}$.

**Proof** Suppose $g \in \mathcal{H}$. Since $z \in \mathcal{I}(\mathcal{H})$ by the hypothesis, $p$ belongs to $\mathcal{I}(\mathcal{H})$ and so $\int_0^z g'(\zeta)p(\zeta)d\zeta \in \mathcal{H}$. Since $p' \in \mathcal{M}(\mathcal{H})$ and $z \in J(\mathcal{H})$, $\int_0^z g(\zeta)p'(\zeta)d\zeta$ belongs to $\mathcal{H}$. Hence $f(z) = \int_0^z (2p'(\zeta)g(\zeta) + p(\zeta)g'(\zeta))d\zeta$ belongs to $\mathcal{H}$. Now the lemma follows because
\[
\int_0^z f'(\zeta)p(\zeta)d\zeta = \int_0^z (p^2(\zeta)g(\zeta))'d\zeta = p^2(z)g(z) + p^2(0)g(0).
\]

**Lemma 5.** Suppose that $B$ is a finite Blaschke product, and both $g$ and $g^{-1}$ are in $\mathcal{I}(\mathcal{H})$. If $\phi = Bg$ then $\phi \in \mathcal{I}(\mathcal{H})$ and $\dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty$.

**Proof** By the hypothesis, $I_B(\mathcal{H}) = I_B(z\mathcal{H}) = I_B(I_g(\mathcal{H})) = I_\phi(\mathcal{H})$ by Lemma 3. We may assume that
\[
B = \prod_{j=1}^n \frac{z - a_j}{1 - \bar{a}_j z} \quad \text{and} \quad \{a_j\} \subset D.
\]
Since $\prod_{j=1}^n (1 - \bar{a}_j z)$ is invertible in $\mathcal{I}(\mathcal{H})$, by Lemma 3 $I_\phi(\mathcal{H}) = I_p(\mathcal{H})$ where $p = \prod_{j=1}^n (z - a_j)$. Lemmas 1 and 4 imply that $\dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty$.

**Lemma 6.** If $p$ is a polynomial then $p(S_z) = S_{p(z)}$.

**Proof** By hypothesis, $P^NI_z(I - P^N) = 0$. Hence
\[
S_{z^2} = P^NI_{z^2}P^N = P^NI_zI_zP^N = P^NI_{z^2}P^N = P^NI_zP^NI_zP^N = S_zS_z.
\]
Now it is easy to see that \( p(S_z) = S_{p(z)} \) for a polynomial \( p \).

**Lemma 7.** If \( M \) is a closed invariant subspace of \( I_z \) and \( \dim \mathcal{H}/M = n < \infty \) then there exists a polynomial \( p \) such that the degree of \( p \leq n \) and \( I_p(\mathcal{H}) \subseteq M \).

**Proof** If we put \( N = \mathcal{H} \ominus M \), then \( \dim N = n < \infty \) and so there exists a polynomial \( p \) such that \( p(S_z) = 0 \) and the degree of \( p \leq n \). By Lemma 6, \( S_{p(z)} = 0 \) and so \( I_p(N) \subseteq M \). Since \( I_p(M) \subseteq M \), \( I_p(\mathcal{H}) \subseteq M \).

**Theorem 2.** Suppose \( \mathcal{I}(\mathcal{H}) \) contains \( H(\overline{D}) \) and if \( f \in \mathcal{I}(\mathcal{H}) \) and \( f(a) = 0 \) for some \( a \in D \) then \( f/(z - a) \) belongs to \( \mathcal{I}(\mathcal{H}) \). \( I_\phi \) is a Fredholm operator on \( \mathcal{H} \) if and only if \( \phi = Bg \) where \( B \) is a finite Blaschke product, and \( g \) and \( g^{-1} \) are in \( \mathcal{I}(\mathcal{H}) \).

**Proof** If \( \phi = Bg \), \( B \) is a finite Blaschke product, \( g \in \mathcal{I}(\mathcal{H}) \) and \( g^{-1} \in \mathcal{I}(\mathcal{H}) \) then by Lemma 5 \( I_\phi(\mathcal{H}) \) is closed and \( \dim \ker I_\phi^* < \infty \). Since \( \ker I_\phi = \mathbb{C} \), index \( I_\phi = 1 - \dim \ker I_\phi^* \) and so \( I_\phi \) is Fredholm. Conversely if \( I_\phi \) is Fredholm then \( I_\phi(\mathcal{H}) \) is closed and \( \dim \mathcal{H}/I_\phi(\mathcal{H}) < \infty \). Since \( I_z I_\phi(\mathcal{H}) \subseteq I_\phi(\mathcal{H}) \), by Lemma 7 there exists a polynomial such that \( I_p(\mathcal{H}) \subseteq I_\phi(\mathcal{H}) \). By Lemma 4 \( I_p(\mathcal{H}) + \mathbb{C} \supset p^2 \mathcal{H} \). Hence there exists a function \( F \) in \( \mathcal{H} \) and \( c \in \mathbb{C} \) such that \( I_\phi(F) + c = p^2 \). Therefore \( F'(z)\phi(z) = 2p(z)p'(z) \) and so the Blaschke part of \( \phi \) is a finite one \( B \). Thus \( \phi \) can be factorized as \( \phi = Bg \) where \( g \in \mathcal{I}(\mathcal{H}) \) and \( g \) has no zeros on \( D \) because \( \mathcal{I}(\mathcal{H}) \) is a subalgebra in \( \mathcal{B}(\mathcal{H}) \) and both \( B \) and \( B^{-1} \) are in \( \mathcal{I}(\mathcal{H}) \). Hence

\[
I_{g^{-1}p}(\mathcal{H}) \subseteq I_{g^{-1}I_\phi(\mathcal{H})} = I_B(\mathcal{H}) \subseteq \mathcal{H}
\]

and so \( g^{-1}p \) belongs to \( \mathcal{I}(\mathcal{H}) \). By hypothesis on \( \mathcal{I}(\mathcal{H}) \), \( g^{-1} \) belongs to \( \mathcal{I}(\mathcal{H}) \).

§ 4. Integral operator \( J_\phi \)

A Fredholm integral operator \( J_\phi \) have not studied. But if \( J_\phi \) is compact then it is not Fredholm. In some special Hilbert space \( \mathcal{H} \), the compactness of \( J_\phi \) have studied.

**Lemma 8.** If \( \phi \) and \( \psi \) are in \( H(D) \) then \( I_\psi J_\phi = J_\phi M_\psi \).
Proof For $f \in \mathcal{H}$

$$
(I_{\phi} J_{\phi} f)(z) = \int_0^z (J_{\phi} f)'(\zeta) \psi(\zeta) d\zeta = \int_0^z f(\zeta) \phi'(\zeta) \psi(\zeta) d\zeta = (J_{\phi} M_{\phi} f)(z)
$$

Lemma 9. If $J_{\phi}$ is a Fredholm operator on $\mathcal{H}$ then $J_{\phi} \mathcal{H}$ is a closed invariant subspace of $I_z$ and $\dim \mathcal{H}/J_{\phi} \mathcal{H} < \infty$. Hence there exists a polynomial $p$ such that $J_{\phi} \mathcal{H} \supset I_p \mathcal{H}$ and so $J_{\phi} \mathcal{H} + \mathbb{C} \supset p^2 \mathcal{H}$.

Proof If $J_{\phi}$ is Fredholm on $\mathcal{H}$ then $J_{\phi} \mathcal{H}$ is a closed subspace and by Lemma 8 $I_z (J_{\phi} \mathcal{H}) \subseteq J_{\phi} \mathcal{H}$. By Lemma 7 there exists a polynomial $q$ such that $I_q \mathcal{H} \subseteq J_{\phi} \mathcal{H}$. Lemma 4 implies this lemma.

Theorem 3. Suppose that there exists a function $g$ in $\mathcal{H}$ such that $g'$ does not belong to $H^2$. Suppose that any function in $\mathcal{H}$ has radial limits almost everywhere. Then there does not exist $J_{\phi}$ which is a Fredholm operator on $\mathcal{H}$.

Proof If $J_{\phi}$ is Fredholm on $\mathcal{H}$ then $J_{\phi} \mathcal{H} + \mathbb{C} \supset p^2 \mathcal{H}$ for some polynomial by Lemma 9. For any $G$ in $p^2 \mathcal{H}$ there exists a function $f$ in $\mathcal{H}$ such that

$$
f(z) \phi'(z) = G'(z) \quad (z \in D).
$$

By hypothesis, there exists $G$ in $p^2 \mathcal{H}$ such that $G' \not\in H^2$ and so $G'$ does not have radial limits on a set of positive measure on $\partial D$ (see [2, Appendix A]). On the other hand, if $G = p^2$ then $G$ has radial limits almost everywhere on $\partial D$. By hypothesis, $f$ has radial limits almost everywhere. This contradiction implies that $J_{\phi}$ is not Fredholm.

§ 5. Relation between $M_{\phi}$ and $I_{\phi}$

Put $Df(z) = f'(z)$ and $J = J_z$, that is, $Jf(z) = \int_0^z f(\zeta) d\zeta$. Then

$$
DJf = f \quad \text{and} \quad JDf = f - f(0).
$$

It is easy to see that $I_{\phi} J = JM_{\phi}$ and $DI_{\phi} = M_{\phi} D$. Put

$$
\mathcal{H}^D = \{ f \in H(D) : Df \in \mathcal{H} \}.
$$
Suppose that $D$ and $J$ are bounded on $\mathcal{H}$ and for $f$ in $\mathcal{H}^D$ put $\|f\|_D^2 = \|Df\|^2 + |f(0)|^2$. Then $\mathcal{H}^D$ is a Hilbert space. Put
\[
\mathcal{H}^J = \{ f \in H(D) : Jf \in \mathcal{H} \}
\]
and for $f$ in $\mathcal{H}^J$ $\|f\|_J = \|Jf\|$. Then $\mathcal{H}^J$ is a Hilbert space.

$D$ is isometric from $\mathcal{H}_0^D = \{ f \in \mathcal{H}^D : f(0) = 0 \}$ onto $\mathcal{H}$. $J$ is isometric from $\mathcal{H}^J$ onto $\mathcal{H}_0^J = \{ f \in \mathcal{H} : f(0) = 0 \}$. Since $DI_\phi = M_\phi D$, $I_\phi$ is bounded on $\mathcal{H}^D$ if and only if $M_\phi$ is bounded on $\mathcal{H}$. Hence $I(\mathcal{H}^D) = M(\mathcal{H})$. Moreover $I_\phi$ is Fredholm on $\mathcal{H}^D$ if and only if $M_\phi$ is Fredholm on $\mathcal{H}$. Since $JM_\phi = I_\phi J$, $I(\mathcal{H}^J) = M(\mathcal{H})$, and $I_\phi$ is Fredholm on $\mathcal{H}^J$ if and only if $M_\phi$ is Fredholm on $\mathcal{H}$. Moreover $(\mathcal{H}^J)^D = (\mathcal{H}^D)^J = \mathcal{H}$. Hence $I(\mathcal{H}) = M(\mathcal{H}^D) = M(\mathcal{H}^J)$, and $I_\phi$ is Fredholm on $\mathcal{H}$ if and only if $M_\phi$ is Fredholm on $\mathcal{H}^D$ and $\mathcal{H}^J$.

§ 6. Examples

Let $dA$ denote the normalized Lebesgue area measure on $D$ and $\omega$ a positive function on $D$ which is summable with respect to $dA$. Put
\[
\mathcal{D}^2(\omega) = \{ f \in H(D) : \|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_D |f'(z)|^2 \omega(z)dA(z) < \infty \}
\]
and
\[
L^2_\omega = \{ f \in H(D) : \|f\|_{L^2}^2 = \int_D |f(z)|^2 \omega(z)dA(z) < \infty \}.
\]
Then $\mathcal{D}^2(\omega)$ is called a weighted Dirichlet space and $L^2_\omega$ is called a weighted Bergman space when $\mathcal{D}^2(\omega)$ and $L^2_\omega$ are nontrivial Hilbert spaces. It is easy to see that $(\mathcal{D}^2(\omega))^J = L^2_\omega$ and $(L^2_\omega)^D = \mathcal{D}^2(\omega)$.

If $\omega(z) = (1 - |z|^2)^\alpha$ and $\alpha > -1$, we will write $\mathcal{D}^2(\omega) = \mathcal{D}^2_\alpha$ and $L^2_\omega = L^2_{\alpha,a}$. It is known that $\mathcal{D}^2_\alpha$ and $L^2_{\alpha,a}$ are nontrivial Hilbert spaces. $\mathcal{D}_1$ is the Hardy space $H^2$, $\mathcal{D}_2$ is the Bergman space $L^2_a$ and $\mathcal{D}_0$ is the Dirichlet space. If $\mathcal{H} = \mathcal{D}_\alpha$ or $L^2_{\alpha,a}$ then $\mathcal{H}$ satisfies the condition $(1)$, $(2)$ and $(3)$ in Introduction. It is known that $H(D) \subset M(\mathcal{D}_\alpha) \subset H^\infty(D)$ and $M(L^2_{\alpha,a}) = H^\infty(D)$. Hence Theorem 1 can apply to $\mathcal{D}_\alpha$ for any $\alpha > -1$. If $\alpha \geq 1$ then $(z-a)\mathcal{D}_\alpha$ is dense in $\mathcal{D}_\alpha$ whenever $a \in \partial D$. Hence Corollary 1 can apply to $\mathcal{D}_\alpha$ for $\alpha \geq 1$. $I(L^2_{\alpha,a}) = M((L^2_{\alpha,a})^D) = M(\mathcal{D}_\alpha)$ and $H(D) \subset M(\mathcal{D}_\alpha) \subset H^\infty(D)$. Since $I(\mathcal{D}_\alpha) = M(L^2_{\alpha,a}) = H^\infty(D)$, Theorem 2 can apply to $\mathcal{D}_\alpha$ for $\alpha > -1$. It is known [3] that $M(\mathcal{D}_\alpha) = H^\infty(D)$ for $\alpha > 1$ and $M(\mathcal{D}_\alpha) = \mathcal{D}_\alpha$ for $-1 < \alpha < 0$. Hence $I(L^2_{\alpha,a}) = H^\infty(D)$ for $\alpha > 1$ and $I(L^2_{\alpha,a}) = \mathcal{D}_\alpha$ for $-1 < \alpha < 0$. Hence Theorem 2 can apply to $L^2_{\alpha,a}$ for $\alpha > 1$ and $-1 < \alpha < 0$. By a theorem in [3],
it is easy to see that $\mathcal{I}(L^2_{a,\alpha}) = \mathcal{M}(\mathcal{D}_\alpha)$ $(0 \leq \alpha \leq 1)$ satisfies the conditions in Theorem 2. Hence Theorem 2 can apply to $L^2_{a,\alpha}$.

When $\mathcal{D}^2(\omega)$ or $L^2_a(\omega)$ is a Hilbert space $\mathcal{H}$, it is important in order to study composition operator that $\mathcal{H}$ satisfies three conditions in Introduction. It will be interesting to determine such a weight $\omega$.

References


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