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Some Fredholm Integration Operators on a Hilbert Space of Holomorphic Functions on the Unit Disc[1]

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Abstract

In this paper, we study when M_ϕ, I_ϕ or J_ϕ is a Fredholm operator on a Hilbert space which satisfies few natural axioms.

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§ 1. Introduction

Let D be the open unit disc in the complex plane $\mathbb{C}$ and $H(D)$ be the set of all analytic functions on D. $H(\bar{D})$ denotes the set of all analytic functions on $\bar{D}$. In this paper, $\mathcal{H}$ is a Hilbert space in $H(D)$ which satisfies the following :

(1) $z\mathcal{H} \subset \mathcal{H}$.
(2) If $a \in D$ then $(z-a)\mathcal{H} \oplus \mathbb{C} = \mathcal{H}$.
(3) $\mathcal{H} \supseteq H(\bar{D})$.

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In this paper, we study the following three operators. If ϕ is a function in $H(D)$, put for $z \in D$,

$$\begin{aligned} (M_\phi f)(z) &= \phi(z)f(z), \\ (I_\phi f)(z) &= \int_0^z f'(\zeta)\phi(\zeta)d\zeta, \\ (J_\phi f)(z) &= \int_0^z f(\zeta)\phi'(\zeta)d\zeta \quad (f \in \mathcal{H}). \end{aligned}$$

Then $(M_\phi f)(z) = (I_\phi f)(z) + (J_\phi f)(z) + \phi(0)f(0)$. It is clear that I_ϕ and J_ϕ are never invertible.

Put $\mathcal{M}(\mathcal{H}) = \{\phi \in H(D) : M_\phi \mathcal{H} \subseteq \mathcal{H}\}$, $\mathcal{I}(\mathcal{H}) = \{\phi \in H(D) : I_\phi \mathcal{H} \subseteq \mathcal{H}\}$ and $\mathcal{J}(\mathcal{H}) = \{\phi \in H(D) : J_\phi \mathcal{H} \subseteq \mathcal{H}\}$. In this paper, we assume that $H(\bar{D}) \subset \mathcal{M}(\mathcal{H})$, $z \in \mathcal{I}(\mathcal{H})$ and $z \in \mathcal{J}(\mathcal{H})$.

§ 2. Multiplication operator M_ϕ

When $\mathcal{M}(\mathcal{H}) = H^\infty(D)$, A. Aleman [1] shows a more general result than Corollary 1 without the condition that $(z-a)\mathcal{H}$ is dense.

Lemma 1. If p is a polynomial with no zeros on ∂D then dim $\mathcal{H}/p\mathcal{H} < \infty$.

Proof If $|a| > 1$ then $(z-a)^{-1} \in H(\bar{D})$ and so $(z-a)^{-1}$ belongs to $M(\mathcal{H})$. Hence we may assume that the zeros of p are contained in D. By hypothesis on $\mathcal{H}$, dim $\mathcal{H}/(z-a)\mathcal{H} = 1$ and so dim $\mathcal{H}/p\mathcal{H} < \infty$.

Lemma 2. If M is a closed invariant subspace of M_z in $\mathcal{H}$ such that dim $\mathcal{H}/M < \infty$, then there exists a polynomial p such that $p\mathcal{H} \subseteq M$.

Proof Let $N = \mathcal{H} \ominus M$ and $S_z = P_N M_z | N$, then S_z is of finite rank because dim $N < \infty$. Hence there exists a polynomial p such that $S_{p(z)} = p(S_z) = 0$. Therefore $pN \subset M$ and so $p\mathcal{H} \subset M$.

Theorem 1.

(1) If $\phi = Bg$ where B is a finite Blaschke product, and both g and g^{-1} are in $\mathcal{M}(\mathcal{H})$ then M_ϕ is a Fredholm operator.

(2) If M_ϕ is a Fredholm operator on $\mathcal{H}$ then $\phi = Bg$ when B is a finite Blaschke product, g is in $\mathcal{M}(\mathcal{H})$ and g^{-1} is in $\mathcal{H}$.

(3) For the g in (2), M_g is a Fredholm operator on $\mathcal{H}$ with index $M_\phi \leq$

index $M_g \leq 0$ and there exists a polynomial q such that $q\mathcal{H} \subseteq g\mathcal{H}$ and the zeros are in $\mathbb{C}\backslash D$.

Proof (1) Suppose $\phi = Bg$, $B = \prod_{j=1}^{n}(z-a_j)/(1-\bar{a}_j z)$, $\{a_j\} \subset D$, and both g and g^{-1} are in $\mathcal{M}(\mathcal{H})$. Since $\mathcal{M}(\mathcal{H}) \supseteq H(\bar{D})$, $\prod_{j=1}^{n}(1-\bar{a}_j z)$ is invertible in $\mathcal{M}(\mathcal{H})$ and so $M_\phi(\mathcal{H}) = p\mathcal{H}$ where $p = \prod_{j=1}^{n}(z-a_j)$.

(2) If M_ϕ is a Fredholm operator then dim $\mathcal{H}/M_\phi(\mathcal{H}) < \infty$ and so by Lemma 2 there exists a polynomial p such that $\phi f = p$. Therefore ϕ can be factorized as $\phi = Bg$ where B is a finite Blaschke product and $g \in \mathcal{H}$. For $\phi \in \mathcal{H}$ and $\prod_{j=1}^{n}(1-\bar{a}_j z)\phi = \prod_{j=1}^{n}(z-a_j)g \in \mathcal{H}$ where $B = \prod_{j=1}^{n}(z-a_j)/(1-\bar{a}_j z)$. Since $Ker\tau_{a_j} = (z-a_j)\mathcal{H}$, g belongs to $\mathcal{H}$. By the similar argument, there exists a function k in $\mathcal{H}$ and $gk = 1$ because $Bgf = p$. Thus g^{-1} belongs to $\mathcal{H}$. We will prove that g belongs to $\mathcal{M}(\mathcal{H})$. Since B is a finite Blaschke product and $Ker\tau_a = (z-a)\mathcal{H}$ for $a \in D$, $\mathcal{H} = K + B\mathcal{H}$ where K is a finite dimensional subspace such that each function in K is a rational function whoes poles are in $\mathbb{C}\backslash\bar{D}$. Since $g \in \mathcal{H}$ and $\mathcal{M}(\mathcal{H}) \supseteq H(\bar{D})$, $gK \subseteq \mathcal{H}$ and so $g\mathcal{H} \subseteq \mathcal{H}$ because $gB\mathcal{H} \subseteq \mathcal{H}$.

(3) By the proof of (2), $p\mathcal{H} \subseteq Bg\mathcal{H} \subseteq g\mathcal{H}$ and so the first statement is clear. Again by the proof of (2), the zeros of p in D is just the zeros of B. This implies that there exists a polynomial q such that $q\mathcal{H} \subseteq g\mathcal{H}$ and q does not have any zeros in D.

Corollary 1. Suppose that $(z-a)\mathcal{H}$ is dense in $\mathcal{H}$ whenever $a \in \partial D$. Then M_ϕ is a Fredholm operator on $\mathcal{H}$ if and only if $\phi = Bg$ where B is a finite Blaschke product, and both g and g^{-1} are in $\mathcal{M}(\mathcal{H})$.

§ 3. Integral operator I_ϕ

It seems to have not been studied yet in this general setting as Theorem 2.

Lemma 3. If ϕ is a function in $\mathcal{I}(\mathcal{H})$ then $I_\phi(\mathcal{H}) = I_\phi(z\mathcal{H}) \subseteq z\mathcal{H}$. $I_\phi(\mathcal{H}) = z\mathcal{H}$ if and only if ϕ and ϕ^{-1} belongs to $\mathcal{I}(\mathcal{H})$.

Proof By the definition of I_ϕ the first statement is clear. We will show the second one. If both ϕ and ϕ^{-1} belong to $\mathcal{I}(\mathcal{H})$, then

$$z\mathcal{H} = I_1(\mathcal{H}) = I_\phi I_{\phi^{-1}}(\mathcal{H}) \subseteq I_\phi(z\mathcal{H}) \subseteq z\mathcal{H}$$

because I_ϕ and $I_{\phi^{-1}}$ are bounded on $\mathcal{H}$. Conversely if $I_\phi(\mathcal{H}) = z\mathcal{H}$ then there exists a function g in $\mathcal{H}$ such that

$$\int_0^z g'(\zeta)\phi(\zeta)d\zeta = z \text{ and so } g'(z)\phi(z) = 1.$$

Hence $\phi^{-1} \in H(D)$ and

$$z\mathcal{H} = I_1(\mathcal{H}) = I_{\phi^{-1}} I_\phi(\mathcal{H}) = I_{\phi^{-1}}(z\mathcal{H})$$

and so both ϕ and ϕ^{-1} belong to $\mathcal{I}(\mathcal{H})$.

Lemma 4. If p is a polynomial then $I_p(\mathcal{H}) + \mathbb{C} \supset p^2\mathcal{H}$.

Proof Suppose $g \in \mathcal{H}$. Since $z \in \mathcal{I}(\mathcal{H})$ by the hypothesis, p belongs to $\mathcal{I}(\mathcal{H})$ and so $\int_0^z g\prime(\zeta)p(\zeta)d\zeta \in \mathcal{H}$. Since $p' \in \mathcal{M}(\mathcal{H})$ and $z \in J(\mathcal{H})$, $\int_0^z g(\zeta)p'(\zeta)d\zeta$ belongs to $\mathcal{H}$. Hence $f(z) = \int_0^z (2p'(\zeta)g(\zeta) + p(\zeta)g'(\zeta))d\zeta$ belongs to $\mathcal{H}$. Now the lemma follows because

$$\int_0^z f'(\zeta)p(\zeta)d\zeta = \int_0^z (p^2(\zeta)g(\zeta))'d\zeta = p^2(z)g(z) + p^2(0)g(0).$$

Lemma 5. Suppose that B is a finite Blaschke product, and both g and g^{-1} are in $\mathcal{I}(\mathcal{H})$. If $\phi = Bg$ then $\phi \in \mathcal{I}(\mathcal{H})$ and dim $\mathcal{H}/I_\phi(\mathcal{H}) < \infty$.

Proof By the hypothesis, $I_B(\mathcal{H}) = I_B(z\mathcal{H}) = I_B(I_g(\mathcal{H})) = I_\phi(\mathcal{H})$ by Lemma 3. We may assume that

$$B = \prod_{j=1}^n \frac{z - a_j}{1 - \bar{a}_j z} \text{ and } \{a_j\} \subset D.$$

Since $\prod_{j=1}^n (1 - \bar{a}_j z)$ is invertible in $\mathcal{I}(\mathcal{H})$, by Lemma 3 $I_\phi(\mathcal{H}) = I_p(\mathcal{H})$ where $p = \prod_{j=1}^n (z - a_j)$. Lemmas 1 and 4 imply that dim $\mathcal{H}/I_\phi(\mathcal{H}) < \infty$.

Lemma 6. If p is a polynomial then $p(S_z) = S_{p(z)}$.

Proof By hypothesis, $P^N I_z (I - P^N) = 0$. Hence

$$\begin{aligned} S_{z^2} &= P^N I_{z^2} P^N = P^N I_z I_z P^N \\ &= P^N I_z (I - P^N) I_z P^N + P^N I_z P^N I_z P^N \\ &= P^N I_z P^N I_z P^N = S_z S_z. \end{aligned}$$

Now it is easy to see that $p(S_z) = S_{p(z)}$ for a polynomial p.

Lemma 7. If M is a closed invariant subspace of I_z and dim $\mathcal{H}/M = n < \infty$ then there exists a polynomial p such that the degree of $p \leq n$ and $I_p(\mathcal{H}) \subseteq M$.

Proof If we put $N = \mathcal{H} \ominus M$, then dim$N = n < \infty$ and so there exists a polynomial p such that $p(S_z) = 0$ and the degree of $p \leq n$. By Lemma 6, $S_{p(z)} = 0$ and so $I_p(N) \subseteq M$. Since $I_p(M) \subseteq M$, $I_p(\mathcal{H}) \subseteq M$.

Theorem 2. Suppose $\mathcal{I}(\mathcal{H})$ contains $H(\bar{D})$ and if $f \in \mathcal{I}(\mathcal{H})$ and $f(a) = 0$ for some $a \in D$ then $f/(z-a)$ belongs to $\mathcal{I}(\mathcal{H})$. I_ϕ is a Fredholm operator on $\mathcal{H}$ if and only if $\phi = Bg$ where B is a finite Blaschke product, and g and g^{-1} are in $\mathcal{I}(\mathcal{H})$.

Proof If $\phi = Bg$, B is a finite Blaschke product, $g \in \mathcal{I}(\mathcal{H})$ and $g^{-1} \in \mathcal{I}(\mathcal{H})$ then by Lemma 5 $I_\phi(\mathcal{H})$ is closed and dim $Ker I_\phi^* < \infty$. Since $Ker I_\phi = \mathbb{C}$, index $I_\phi = 1 - \dim Ker I_\phi^*$ and so I_ϕ is Fredholm. Conversely if I_ϕ is Fredholm then $I_\phi(\mathcal{H})$ is closed and dim $\mathcal{H}/I_\phi(\mathcal{H}) < \infty$. Since $I_z I_\phi(\mathcal{H}) \subseteq I_\phi(\mathcal{H})$, by Lemma 7 there exists a polynomial such that $I_p(\mathcal{H}) \subseteq I_\phi(\mathcal{H})$. By Lemma 4 $I_p(\mathcal{H}) + \mathbb{C} \supset p^2\mathcal{H}$. Hence there exists a function F in $\mathcal{H}$ and $c \in \mathbb{C}$ such that $I_\phi(F) + c = p^2$. Therefore $F'(z)\phi(z) = 2p(z)p'(z)$ and so the Blaschke part of ϕ is a finite one B. Thus ϕ can be factorized as $\phi = Bg$ where $g \in \mathcal{I}(\mathcal{H})$ and g has no zeros on D because $\mathcal{I}(\mathcal{H})$ is a subalgebra in $\mathcal{B}(\mathcal{H})$ and both B and B^{-1} are in $\mathcal{I}(\mathcal{H})$. Hence

$$I_{g^{-1}p}(\mathcal{H}) \subseteq I_{g^{-1}} I_\phi(\mathcal{H}) = I_B(\mathcal{H}) \subseteq \mathcal{H}$$

and so $g^{-1}p$ belongs to $\mathcal{I}(\mathcal{H})$. By hypothesis on $\mathcal{I}(\mathcal{H})$, g^{-1} belongs to $\mathcal{I}(\mathcal{H})$.

§ 4. Integral operator J_ϕ

A Fredholm integral operator J_ϕ have not studied. But if J_ϕ is compact then it is not Fredholm. In some special Hilbert space $\mathcal{H}$, the compactness of J_ϕ have studied.

Lemma 8. If ϕ and ψ are in $H(D)$ then $I_\psi J_\phi = J_\phi M_\psi$.

Proof For $f \in \mathcal{H}$

$$\begin{aligned}(I_\psi J_\phi f)(z) &= \int_0^z (J_\phi f)'(\zeta)\psi(\zeta)d\zeta = \int_0^z f(\zeta)\phi'(\zeta)\psi(\zeta)d\zeta \\ &= (J_\phi M_\psi f)(z)\end{aligned}$$

Lemma 9. If J_ϕ is a Fredholm operator on $\mathcal{H}$ then $J_\phi\mathcal{H}$ is a closed invariant subspace of I_z and dim $\mathcal{H}/J_\phi\mathcal{H} < \infty$. Hence there exists a polynomial p such that $J_\phi\mathcal{H} \supseteq I_p\mathcal{H}$ and so $J_\phi\mathcal{H} + \mathbb{C} \supseteq p^2\mathcal{H}$.

Proof If J_ϕ is Fredholm on $\mathcal{H}$ then $J_\phi\mathcal{H}$ is a closed subspace and by Lemma 8 $I_z(J_\phi\mathcal{H}) \subseteq J_\phi\mathcal{H}$. By Lemma 7 there exists a polynomial q such that $I_q\mathcal{H} \subseteq J_\phi\mathcal{H}$. Lemma 4 implies this lemma.

Theorem 3. Suppose that there exists a function g in $\mathcal{H}$ such that g' does not belong to H^2. Suppose that any function in $\mathcal{H}$ has radial limits almost everywhere. Then there does not exist J_ϕ which is a Fredholm operator on $\mathcal{H}$.

Proof If J_ϕ is Fredholm on $\mathcal{H}$ then $J_\phi\mathcal{H} + \mathbb{C} \supseteq p^2\mathcal{H}$ for some polynomial by Lemma 9. For any G in $p^2\mathcal{H}$ there exists a function f in $\mathcal{H}$ such that

$$f(z)\phi'(z) = G'(z) \quad (z \in D).$$

By hypothesis, there exists G in $p^2\mathcal{H}$ such that $G' \notin H^2$ and so G' does not have radial limits on a set of positive measure on ∂D (see [2, Appendix A]). On the other hand, if $G = p^2$ then G has radial limits almost everywhere on ∂D. By hypothesis, f has radial limits almost everywhere. This contradiction implies that J_ϕ is not Fredholm.

§ 5. Relation between M_ϕ and I_ϕ

Put $Df(z) = f'(z)$ and $J = J_z$, that is, $Jf(z) = \int_0^z f(\zeta)d\zeta$. Then

$$DJf = f \text{ and } JDf = f - f(0).$$

It is easy to see that $I_\phi J = JM_\phi$ and $DI_\phi = M_\phi D$. Put

$$\mathcal{H}^D = \{f \in H(D) : Df \in \mathcal{H}\}$$

Suppose that D and J are bounded on $\mathcal{H}$ and for f in $\mathcal{H}^D$ put $\|f\|_D^2 = \|Df\|^2 + |f(0)|^2$. Then $\mathcal{H}^D$ is a Hilbert space. Put

$$\mathcal{H}^J = \{f \in H(D) : Jf \in \mathcal{H}\}$$

and for f in $\mathcal{H}^J$ $\|f\|_J = \|Jf\|$. Then $\mathcal{H}^J$ is a Hilbert space.

D is isometric from $\mathcal{H}_0^D = \{f \in \mathcal{H}^D : f(0) = 0\}$ onto $\mathcal{H}$. J is isometric from $\mathcal{H}^J$ onto $\mathcal{H}_0 = \{f \in \mathcal{H} : f(0) = 0\}$. Since $DI_\phi = M_\phi D$, I_ϕ is bounded on $\mathcal{H}^D$ if and only if M_ϕ is bounded on $\mathcal{H}$. Hence $\mathcal{I}(\mathcal{H}^D) = \mathcal{M}(\mathcal{H})$. Moreover I_ϕ is Fredholm on $\mathcal{H}^D$ if and only if M_ϕ is Fredholm on $\mathcal{H}$. Since $JM_\phi = I_\phi J$, $\mathcal{I}(\mathcal{H}^J) = \mathcal{M}(\mathcal{H})$, and I_ϕ is Fredholm on $\mathcal{H}^J$ if and only if M_ϕ is Fredholm on $\mathcal{H}$. Moreover $(\mathcal{H}^J)^D = (\mathcal{H}^D)^J = \mathcal{H}$. Hence $\mathcal{I}(\mathcal{H}) = \mathcal{M}(\mathcal{H}^D) = \mathcal{M}(\mathcal{H}^J)$, and I_ϕ is Fredholm on $\mathcal{H}$ if and only if M_ϕ is Fredholm on $\mathcal{H}^D$ and $\mathcal{H}^J$.

§ **6. Examples**

Let dA denote the normalized Lebesgue area measure on D and ω a positive function on D which is summable with respect to dA. Put

$$\mathcal{D}^2(\omega) = \{f \in H(D) : \|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_D |f'(z)|^2 \omega(z) dA(z) < \infty\}$$

and

$$L_a^2(\omega) = \{f \in H(D) : \|f\|_{L_a^2}^2 = \int_D |f(z)|^2 \omega(z) dA(z) < \infty\}.$$

Then $\mathcal{D}^2(\omega)$ is called a weighted Dirichlet space and $L_a^2(\omega)$ is called a weighted Bergman space when $\mathcal{D}^2(\omega)$ and $L_a^2(\omega)$ are nontrivial Hilbert spaces. It is easy to see that $(\mathcal{D}^2(\omega))^J = L_a^2(\omega)$ and $(L_a^2(\omega))^D = \mathcal{D}^2(\omega)$.

If $\omega(z) = (1 - |z|^2)^\alpha$ and $\alpha > -1$, we will write $\mathcal{D}^2(\omega) = \mathcal{D}_\alpha^2$ and $L_a^2(\omega) = L_{a,\alpha}^2$. It is known that $\mathcal{D}_\alpha^2$ and $L_{a,\alpha}^2$ are nontrivial Hilbert spaces. $\mathcal{D}_1$ is the Hardy space H^2, $\mathcal{D}_2$ is the Bergman space L_a^2 and $\mathcal{D}_0$ is the Dirichlet space. If $\mathcal{H} = \mathcal{D}_\alpha$ or $L_{a,\alpha}^2$ then $\mathcal{H}$ satisfies the condition (1), (2) and (3) in Introduction. It is known that $H(\bar{D}) \subset \mathcal{M}(\mathcal{D}_\alpha) \subset H^\infty(D)$ and $\mathcal{M}(L_{a,\alpha}^2) = H^\infty(D)$. Hence Theorem 1 can apply to $\mathcal{D}_\alpha$ for any $\alpha > -1$. If $\alpha \geq 1$ then $(z - a)\mathcal{D}_\alpha$ is dense in $\mathcal{D}_\alpha$ whenever $a \in \partial D$. Hence Corollary 1 can apply to $\mathcal{D}_\alpha$ for $\alpha \geq 1$. $\mathcal{I}(L_{a,\alpha}^2) = \mathcal{M}((L_{a,\alpha}^2)^D) = \mathcal{M}(\mathcal{D}_\alpha)$ and $H(\bar{D}) \subset \mathcal{M}(\mathcal{D}_\alpha) \subset H^\infty(D)$. Since $\mathcal{I}(\mathcal{D}_\alpha) = \mathcal{M}(L_{a,\alpha}^2) = H^\infty(D)$, Theorem 2 can apply to $\mathcal{D}_\alpha$ for $\alpha > -1$. It is known [3] that $\mathcal{M}(\mathcal{D}_\alpha) = H^\infty(D)$ for $\alpha > 1$ and $\mathcal{M}(\mathcal{D}_\alpha) = \mathcal{D}_\alpha$ for $-1 < \alpha < 0$. Hence $\mathcal{I}(L_{a,\alpha}^2) = H^\infty(D)$ for $\alpha > 1$ and $\mathcal{I}(L_{a,\alpha}^2) = \mathcal{D}_\alpha$ for $-1 < \alpha < 0$. Hence Theorem 2 can apply to $L_{a,\alpha}^2$ for $\alpha > 1$ and $-1 < \alpha < 0$. By a theorem in [3], it is easy to see that $\mathcal{I}(L^2_{a,\alpha}) = \mathcal{M}(\mathcal{D}_\alpha)$ $(0 \leq \alpha \leq 1)$ satisfies the conditions in Theorem 2. Hence Theorem 2 can apply to $L^2_{a,\alpha}$.

When $\mathcal{D}^2(\omega)$ or $L^2_a(\omega)$ is a Hilbert space $\mathcal{H}$, it is important in order to study composition operator that $\mathcal{H}$ satisfies three conditions in Introduction. It will be interesting to determine such a weight ω.

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