

On M-Spaces and Banach Spaces

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Abstract

We define in this paper the concept of C-space, related with M-spaces and Banach spaces. We obtain various properties on these spaces and propose some open problems.

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1 Introduction

There exist three causes that motive this new paper. First, an early theorem of Corson, also the concept of M -space (defined by K.Morita), and finally a paper on Banach spaces by the author:

The Corson Theorem.[3] *For any covering \mathcal{U} of a infinite dimensional reflexive Banach space B , where \mathcal{U} is formed by bounded, convex sets, there is not a point x in B such that each neighborhood of x meets only finitely many members of \mathcal{U} , i.e., \mathcal{U} is not locally finite.*

In our paper [6], we study some problems related to the Corson Theorem. In particular, we proved that: "For every $r \geq 0$, there exists an open covering of c_0 , which is locally finite and is formed by balls of radius r ".

We will use in this paper the concept of M -space:

Definition 1. [7] A paracompact space X is called a M -space if there is some perfect map from X onto some metric space.

2 Main results.

Definition 2. Let X be a topological space. We will say that X is a C -space if there is some Banach space E and some perfect map f from X onto E such that exists a locally finite covering of X formed by pre-images of open balls of radius 1 by the map f .

- Remarks. 1.** If X is a C -space then X is a paracompact M -space.
2. c_0, E_∞, IR^n are C -spaces.

Proposition 1. Let X be a topological space, E be a Banach space and f be a perfect map from X onto E . Then

$\mathcal{V} = \{f^{-1}(B_1(x_j)) | j \in J\}$ is a locally finite covering of X , if and, only if $\{B_1(x_j) | j \in J\}$ is a locally finite covering of E .

Proof. (\Rightarrow) If \mathcal{V} covers X also $\{B_1(x_j) | j \in J\}$ covers E , because f is onto.

For each $z \in E$ and each $x \in f^{-1}(z)$ there exists an open neighborhood U_z^x of x , such that meets only finitely members of \mathcal{V} . Then $\{U_z^x | x \in f^{-1}(z)\}$ is an open covering of $f^{-1}(z)$, and $f^{-1}(z) \subset \cup_{k=1}^r U_z^{x_k}$ (for some $x_1, \dots, x_r \in f^{-1}(z)$) because f is a perfect map.

Since f is closed, there exists an open neighborhood W^z of z such that $f^{-1}(W^z) \subset \cup_{k=1}^r U_z^{x_k}$.

Then, $f^{-1}(W^z)$ meets only finitely members of \mathcal{V} , and also W^z meets only finitely members of $\{B_1(x_j) | j \in J\}$.

(\Leftarrow) If $\{B_1(x_j) | j \in J\}$ covers E , then \mathcal{V} covers X .

For each $x \in X$ there exists an open neighborhood $V^{f(x)}$ of $f(x)$ such that meets only finitely members of $\{B_1(x_j) | j \in J\}$. Clearly, $f^{-1}(V^{f(x)})$ is an open neighborhood of x and meets only finitely members of \mathcal{V} .

Corollary 1. Let X be a topological space. Then, X is a C -space if and only if there exists a Banach space E such that has a locally finite open covering formed by balls of fixed radius, and a perfect map f from X onto E .

Corollary 2. For each compact space K , we have that $c_0 \times K$ is a C -space.

Proof. Since the projection map p_1 is a perfect map from $c_0 \times K$ onto c_0 .

Corollary 3. For each compact space K , we have that $IR^N \times K$ is a C -space.

Proof. It follows from the above Corollary, because c_0 is homeomorphic to IR^{IN} (theorems of Kadec [5] and Anderson [1]).

Proposition 2. *Let X be a topological space. If X is separable and C -space, then it is homeomorphic to some closed subset of $IR^{IN} \times I^{IN}$.*

Proof. Since X is a separable C -space, there is some separable Banach space E and some perfect map from X onto E . From [8, Theorem 2] it follows that X is homeomorphic to a closed subset in $E \times I^{IN}$. Finally, theorems of Kadec [5] and Anderson [1] yield the conclusion.

3 Open problems.

1. Let X be a topological space. Have we that X is a C -space if and only if X is homeomorphic to a closed subset of $IR^{IN} \times I^{w(X)}$? (where $w(X)$ is the weight of X).

2. Have the normed spaces with locally finite coverings by balls analogous properties to totally bounded spaces?

4 References

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