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# A Linear Model for MRG and Quaternion Operator

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**Abstract.** The first thing MRG does functionally is to make the axis of atoms which form the mass studied parallel to the existing axis. In this study,a linear model has been developed for the transformation of axis which parallels the axis to an existing direction.

## 1. REAL QUATERNIONS

1.1. **Introduction.** Quaternions were first of all recognized in 1843 by William Rowan Hamilton(1805-1865). With the recognition of quaternions by Hamilton,operation of the division process become possible for the two vectors as well. Therefore, a new multiplication process was added to the vector algebra and the analysis of motions in 3-dimensional space became facilitated.

1.2. **The Algebra Of Real Quaternions.** A real quaternion can be defined by the accompanying of ordinal four numbers with four units like $+1$, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$. Here the first unit $+1$ is a real number, the other three units possess the particularities of

$$\begin{aligned}
&\vec{e}_1^2 = \vec{e}_2^2 = \vec{e}_3^2 = -1 \\
&\vec{e}_1 \times \vec{e}_2 = \vec{e}_3, \vec{e}_2 \times \vec{e}_3 = \vec{e}_1, \vec{e}_3 \times \vec{e}_1 = \vec{e}_2 \\
&\vec{e}_2 \times \vec{e}_1 = -\vec{e}_3, \vec{e}_3 \times \vec{e}_2 = -\vec{e}_1, \vec{e}_1 \times \vec{e}_3 = -\vec{e}_2
\end{aligned}$$

Hence a quaternion can be expressed as $q = d + a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$ Here the $d,\ a,\ b,\ c \in R$ real numbers are termed as q quaternion components.$\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ units can be taken as the base vectors of a vertical coordinate system of areal vectors of a vertical coordinate system of areal vector space with 3-dimensions,and therefore the q quaternion can be divided into two sections:one is the scalar section shown with $S_q$ and the other is the vectoral section shown with $\vec{V}_q$.

$$S_q = d, \quad \vec{V}_q = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$$

hence

$$q = S_q + \vec{V}_q.$$

1.3. **Unit Quaternion And Norming.** A quaternion is termed as unit quaternion if its norm possesses a unit, thus it is shown as $q_0$. Thus the norming of any quaternion can be expressed as

$$q_0 = \frac{q}{\sqrt{N_q}} = \frac{d + a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3}{\sqrt{d^2 + a^2 + b^2 + c^2}}$$

This $q_0$ unit quaternion can be written in the form of ,

$$q_0 = \cos\theta + \vec{S}_0 \sin\theta$$

Here, $\cos\theta = \frac{d}{\sqrt{d^2+a^2+b^2+c^2}}$, $\sin\theta = \frac{a^2+b^2+c^2}{\sqrt{d^2+a^2+b^2+c^2}}$ and when $a^2 + b^2 + c^2 \neq 0$ the $\vec{S}_0 = \frac{a\vec{e}_1+b\vec{e}_2+c\vec{e}_3}{\sqrt{a^2+b^2+c^2}}$. unit vector is called $q_0$ unit quaternion axis.

1.4. **The Quaternion Multiplication of two vectors.** Let $p, q \in Q$ hence the quaternion multiplication can be given with the equality of $p \otimes q = [S_p, V_p] \otimes [S_q, V_q] = S_p.S_q - \langle V_p, V_q \rangle + S_p.V_q + S_q.V_p + V_p \times V_q$ (Siminovitch, 1995) if this multiplication is written in the form of the components and provided that $p = (p_0, p_1, p_2, p_3)$ and $q = (q_0, q_1, q_2, q_3)$ then we can obtain

$$p \otimes q = (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3, (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2)\vec{i}$$
$$+(p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1)\vec{j} + (p_0q_3 - p_1q_2 - p_2q_1 + p_2q_0)\vec{k}$$

(Siminovitch,1995)

## 2. Magnetic Resonance Imaging

2.1. **Introduction.** The magnetic moment of particles in a nucleus revolving around itself is constituted parallelled to the rotation axis

This magnetic movement caused by protons is directly associated with the rotation of protons around themselves,this rotation movement is called spin movement.

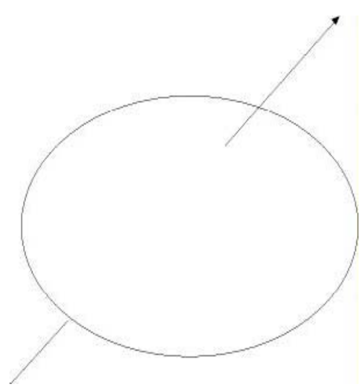

Figure 1. The spin movement of protons

The magnetic moment is defined with the vectoral system showing the power and direction of magnetic field protons in a powerful magnetic field reach to a lower or higher energy levels. In the other words, some parts of protons show a sequencing paralleled to magnet vector where as other parts show an anti-parallel (opposite) sequencing. Those sequenced in a paralleled way have lower energy levels and those sequenced in an opposite (anti-parallel) direction are characterized as having higher energy levels. The numbers of these protons are not equal. The magnitude of the difference in between depends on the power of the magnet $\vec{B}_0$. The more the power of the magnet, the greater the difference in between (the number of those showing parallel sequencing increases) With any minute disparity in favor of this paralleled sequencing, the magnetic vector of this bulk is formed and here MR (Magnetic Resonance) image is obtained.

2.2. **Resonance.** The net magnetic vector formed in the bulk placed in the powerful magnetic field is paralleled to the $\vec{B}_0$ vector. In this case, radio frequency should be used in order to obtain signals from the bulk through the use of net magnetic vector. The Exchange of positions of some protons making emission movement by absorbing energy from the radio wave and returning of the same protons to their original positions by emitting the absorbed energy to the environment is called resonance.

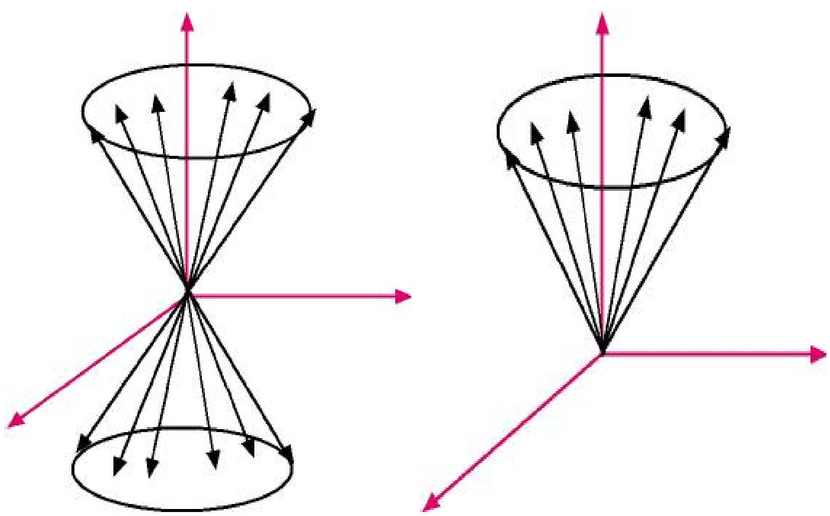

FIGURE 2. a) The resonance movement of a proton b) The resonance movement of about axis z

2.3. **Emission Movement.** Considering a single proton, it makes a movement similar to the emission movement of a topage(ball)around $\vec{B}_0$vector as well as revolving around itself and the movement around $\vec{B}_0$ is called emission moment.

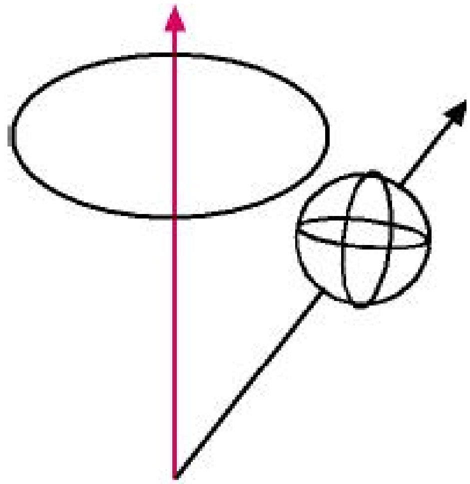

FIGURE 3. The emission Movement of protons

## 3. ROTATION OPERATOR

In this study a linear model will be studied for magnetic resonance. Quaternion operator will be used as an instrument.

The most preliminary function in the structure of magnetic resonance is to make the rotation axis of nucleus paralleled to a determined axis

Let z axis be a determined one with the supposition of Euclid coordinate system in$R^3$and its determined vector be $\vec{b} = (0, 0, 1)$

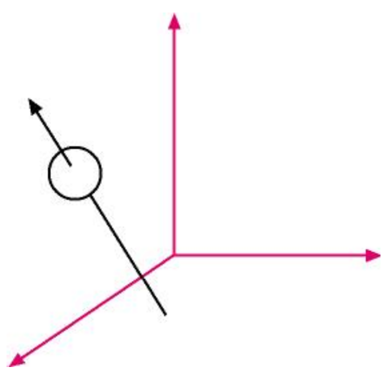

FIGURE 4. The Position of Quaternion Operator

Hence the problem develops into the form of $\vec{X}_p$ vector's operator evolving into $\vec{x}_p$ vector paralleled to $\vec{b}$.

The norm of the plane determined by $\vec{X}_p$ and $\vec{b}$ vectors is $\vec{n} = \frac{\vec{x}\times\vec{b}}{\|\vec{x}\|\|\vec{b}\|}$ and the angel between them if calculated clearly:

$$\vec{n} = \frac{\vec{x}\times\vec{b}}{\|\vec{x}\|\left\|\vec{b}\right\|\sin\theta} = \frac{\begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ 0 & 0 & 1 \end{vmatrix}}{\sqrt{x_1^2+x_2^2+x_3^2.\sqrt{0^2+0^2+1^2\sin\theta}}}$$

$$= \frac{(x_2,-x_1,0)}{\sqrt{x_1^2+x_2^2+x_3^2}\sin\theta}$$

$$\theta = \arccos\frac{\langle\vec{x},\vec{b}\rangle}{\|\vec{x}\|\left\|\vec{b}\right\|} = \arccos\frac{\langle(x_1,x_2,x_3),(0,0,1)\rangle}{\sqrt{x_1^2+x_2^2+x_3^2}\sqrt{0^2+0^2+1^2}}$$

$$.\arccos\frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}$$

$$\theta = \arccos\frac{<\overrightarrow{x},\overrightarrow{b}>}{\|\overrightarrow{x}\|\|\overrightarrow{b}\|}$$

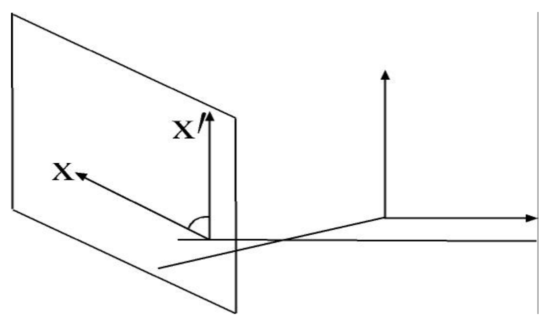


FIGURE 5. The Position of Parallelism Operator

Provided that selections and notations are designed in this way, this theorem could be stated

**Theorem 3.1.** *Provided that*

$$\vec{b} = (0,0,1), \vec{x} = (x_1, x_2, x_3) \neq \vec{b} \quad , \theta = angle(\vec{x}, \vec{b})$$

*and*

$$\vec{n} = \frac{\vec{x} \times \vec{b}}{\|\vec{x}\| \left\|\vec{b}\right\| \sin\theta} = \frac{(x_2, -x_1, 0)}{\sqrt{x_1^2 + x_2^2 + x_3^2}\frac{\sqrt{x_1^2+x_2^2}}{\sqrt{x_1^2+x_2^2+x_3^2}}}$$

*and*

$$Q = \cos\theta + \vec{n}\sin\theta$$

*be under a determined quaternion operator we take*

$$\vec{x}' // \vec{b} \text{ for } Q(\vec{x}) = \vec{x}'$$

.

*Proof.*

$$\vec{n} = \frac{\vec{x} \times \vec{b}}{\|\vec{x}\| \left\|\vec{b}\right\| \sin\theta} = \frac{(x_2, -x_1, 0)}{\sqrt{x_1^2 + x_2^2}}$$

$$\theta = \arccos \frac{\langle \vec{x}, \vec{b} \rangle}{\|\vec{x}\| \left\|\vec{b}\right\|} = \arccos \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\cos\theta = \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \quad , \quad \sin\theta = \frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\begin{aligned}
Q &= \cos\theta + \vec{n}\sin\theta \\
&= \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} + \frac{(x_2, -x_1, 0)}{\sqrt{x_1^2 + x_2^2}} . \frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\
&= \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} + \frac{(x_2, -x_1, 0)}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\
&= \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} + \left(\frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \frac{-x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, 0\right)
\end{aligned}$$

$$
\begin{aligned}
Q \otimes \vec{x} =& S_q.S_x - \langle V_q, V_x \rangle + S_q.V_x + S_x.V_q + V_q \times V_x \\
=& \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}.0 + \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}.(x_1, x_2, x_3) \\
& + 0.\left(\frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{-x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}, 0\right) \\
& - \left\langle \left(\frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{-x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}, 0\right), (x_1, x_2, x_3) \right\rangle \\
& + \begin{vmatrix} i & j & k \\ \frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}} & \frac{-x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} & 0 \\ x_1 & x_2 & x_3 \end{vmatrix} \\
=& \left( \frac{x_1 x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{x_2.x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{x_3 x_3}{\sqrt{x_1^2+x_2^2+x_3^2}} \right) \\
& - \left( \frac{x_2 x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} - \frac{x_1 x_2}{\sqrt{x_1^2+x_2^2+x_3^2}} \right) \\
& + \left( \frac{-x_3 x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{-x_2 x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{x_2^2}{\sqrt{x_1^2+x_2^2+x_3^2}} \right. \\
& \left. + \frac{x_1^2}{\sqrt{x_1^2+x_2^2+x_3^2}} \right) \\
=& \left( 0, 0, \frac{x_1^2+x_2^2+x_3^2}{\sqrt{x_1^2+x_2^2+x_3^2}} \right) = \lambda \vec{b}
\end{aligned}
$$

□

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